On the Jacobian polygon and Łojasiewicz exponent of isolated complex hypersurface singularities

In commemoration of the centenary of Professor Stanisław Łojasiewicz

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The Łojasiewicz exponent

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ, i.e.

$$f \in \mathbb{C}\{x_0, x_1, \ldots, x_n\}.$$

We will assume that f has an isolated singularity at 0.

Definition

The *Łojasiewicz exponent of* f $\mathcal{L}(f, 0)$ is the smallest $\theta > 0$ such that there exists a $C > 0$ such that near 0,

$$\|x\|^{\theta} \leq C\|\nabla f(x)\|.$$

Example

If $f(x) = \sum_i x_i^2$ is a Morse-point, then $\nabla f(x) = 2x$. As a result, $\|x\| = \frac{1}{2}\|\nabla f(x)\|$, and so $\mathcal{L}(f, 0) = 1$.

What is this paper about?

- Arnol'd suggests:

  Every interesting discrete invariant of a generic singularity with Newton polyhedron Γ *is an interesting function of the polyhedron.*

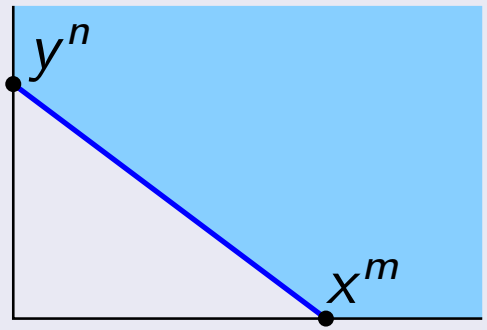
- In turn, Brzostowski, Krasiński and Oleksik conjectured a formula for the Łojasiewicz exponent associated with an isolated Newton nondegenerate function, in terms of its Newton polyhedron.
- My attempts to prove this conjecture lead to a counterexample.
- The conjecture requires a small, but unknown, modification to hold true.
- I found a more complicated formula for $\mathcal{L}(f,0)$ in terms of $\Gamma_+(f)$.
- Along the way, I found some results on Teissier's *vanishing rates* or *«taux d'évanescence»*.

Newton polygons

Definition

The set of *Newton polygons* in $\mathbb{R}^2_{\geq 0}$ form a cancellative semigroup $\mathfrak{N}$ under Minkowski sum. Denote by $K\mathfrak{N}$ its group of differences. Either is generated by elements of the form

$$\left\{ \frac{m}{n} \right\}$$



with $m, n \in \mathbb{Z}_{>0} \cup \{\infty\}$, and relations

$$\left\{ \frac{cm}{cn} \right\} = c \left\{ \frac{m}{n} \right\}, \qquad c \in \mathbb{Z}_{>0}.$$

Newton polygons

Thus, any element $N \in K\mathfrak{N}$ can be written uniqely as

$$N = \sum_{\alpha=m/n} a_\alpha \left\{ \frac{m}{\overline{\overline{n}}} \right\}$$

where m/n are reduced fractions.

Definition

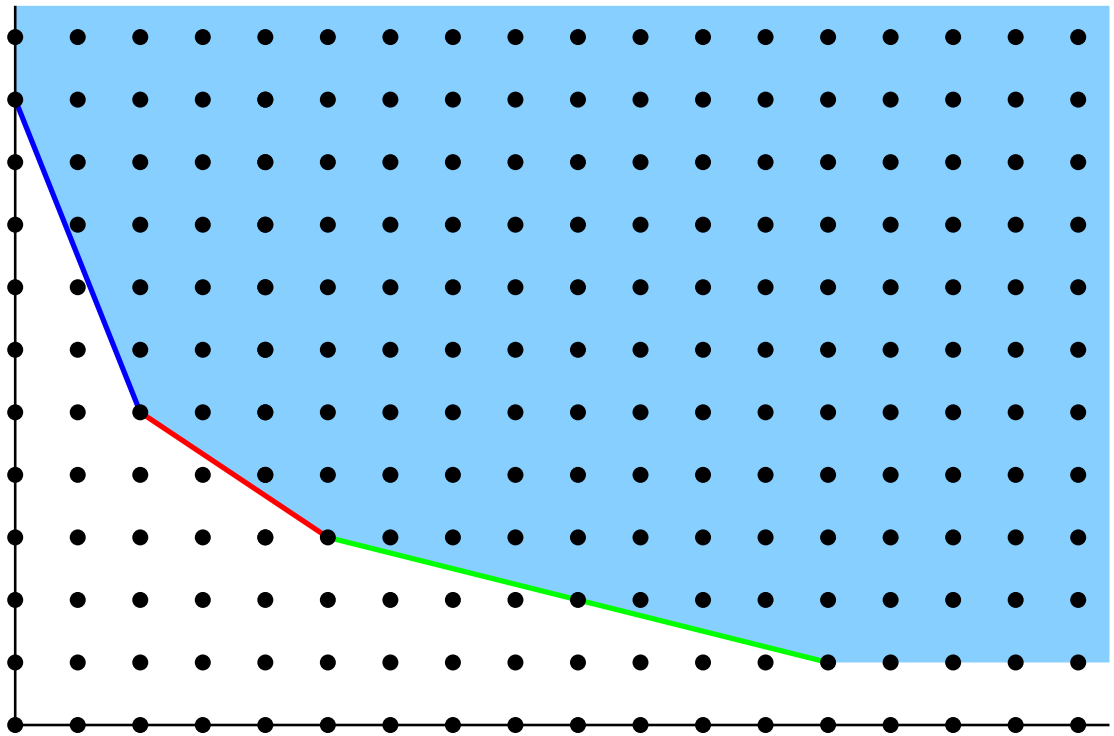
The support of N is

$$\operatorname{supp}(N) = \{\alpha \in \mathbb{Q}_{\geq 0} \cup \{\infty\} \mid a_\alpha \neq 0\}$$

The *degree* of $N \in K\mathfrak{N}$ is the largest fraction m/n for $\{m//n\}$ appearing with nonzero coefficient,

$$\deg(N) = \max \operatorname{supp}(N).$$

Example



$$\left\{\frac{2}{5}\right\} + \left\{\frac{3}{2}\right\} + \left\{\frac{8}{2}\right\} + \left\{\frac{\infty}{1}\right\}$$

The polar curve

Take generic coordinates $x_0, \ldots, x_n$ and consider the polar curve

$$P = \left\{ \frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \ldots = \frac{\partial f}{\partial x_{n-1}} \right\} \subset \mathbb{C}^{n+1}$$

with respect to the hyperplane

$$H = \{x_n = 0\}.$$

It is a union of reduced branches

$$P = P_1 \cup P_2 \cup \ldots \cup P_l.$$

Teissier's Jacobian polygon

Each branch defines numerical data

$$m_q = (X, P_q)_{\mathbb{C}^{n+1},0}\,, \qquad n_q = (H, P_q)_{\mathbb{C}^{n+1},0}\,, \qquad \alpha_q = \frac{m_q}{n_q}.$$

Definition (Teissier '77)

The Jacobian polygon is defined as

$$\mathrm{J}(f,0) = \sum_{q=1}^{l} \left\{ \frac{m_q}{n_q} \right\} \in \mathfrak{N}\,.$$

Theorem (Teissier '77)

The Jacobian polygon determines the Łojasiewicz exponent

$$\mathcal{L}(f,0) = \deg(\mathrm{J}(f,0)) - 1.$$

The alternating Jacobian polygon

Definition

Denote by $f^{(d+1)}(x_0, \ldots, x_d)$ the restriction of f to a generic $(d+1)$-linear subspace

$$H^{(d+1)} = \{x_{d+1} = \cdots = x_n = 0\}.$$

Definition

The alternating Jacobian polygon is

$$\mathrm{AJ}(f, 0) = \sum_{d=0}^{n} (-1)^{n-d} \, \mathrm{J}(f^{(d+1)}, 0) \in K\mathfrak{N}.$$

The alternating Jacobian polygon

Theorem (-)

The alternating Jacobian polygon determines the Łojasiewicz exponent

$$\mathcal{L}(f,0) = \deg(\mathrm{AJ}(f,0)) - 1$$

except when n is odd and f is a Morse point.

An A'Campo style formula

Assume given an embedded resolution of f

$$\pi : (Y, D) \to (\mathbb{C}^{n+1}, 0).$$

The exceptional divisor decomposes as $D = \cup_{i \in I} D_i$, and we set

$$D_i^\circ = D_i \setminus (\cup_{j \neq i} D_j)\,.$$

Theorem (-)

The alternating Jacobian polygon is determined by π

$$\mathrm{AJ}(f, 0) = (-1)^n \sum_{i \in I} \chi\left(D_i^\circ \setminus \tilde{X}\right) \left\{ \frac{m_i}{n_i} \right\}$$

where $\tilde{X}$ is the strict transform of X, and

$$m_i = \mathrm{ord}_{D_i}(f), \qquad n_i = \mathrm{ord}_{D_i}(x_n).$$

Newton diagram and coordinate facets

Assume that $f \in \mathbb{C}\{z_0, z_1, \ldots, z_n\}$ is Newton nondegenerate, and has Newton diagram $\Gamma(f)$.

Definition

A *facet* $F \subset \Gamma(f)$ is a face of dimension n. It has a *primitive normal vector* $v_F \in \mathbb{Z}_{>0}^{n+1}$ such that

$$F \subset \left\{ u \in \mathbb{R}^{n+1} \,\middle|\, \langle u, v_F \rangle = m_F \right\}, \qquad m_F \in \mathbb{Z}_{\geq 0}$$

Define n_F as the smallest coordinate of v_F

$$n_F = \min\{v_0, v_1, \ldots, v_n\} \qquad \text{if} \qquad v_F = (v_0, v_1, \ldots, v_n).$$

A *coordinate facet* is a facet of the diagram $\Gamma(f) \cap \mathbb{R}^I$ for some $I \subset \{0, \ldots, n\}$.

A Varchenko style formula

Theorem (-)

The Newton diagram determines the alternating Jacobian polygon

$$\mathrm{AJ}(f,0) = \sum_F (-1)^{n-s} s!\,\mathrm{Vol}_s\, F \left\{ \frac{m_F}{n_F} \right\}.$$

where the sum runs through all coordinate facets of $\Gamma(f)$*, and* $s = \dim F$*.*

Corollary (Kouchnirenko '76)

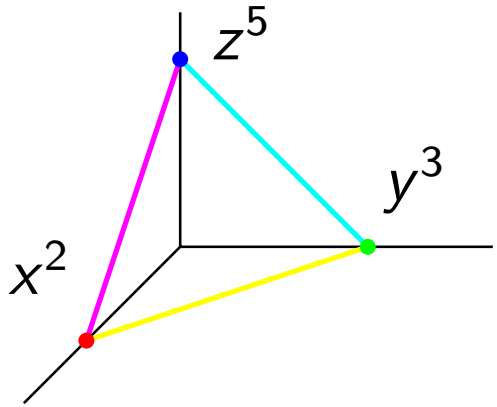
Newton diagram determines the Milnor number via the Newton number

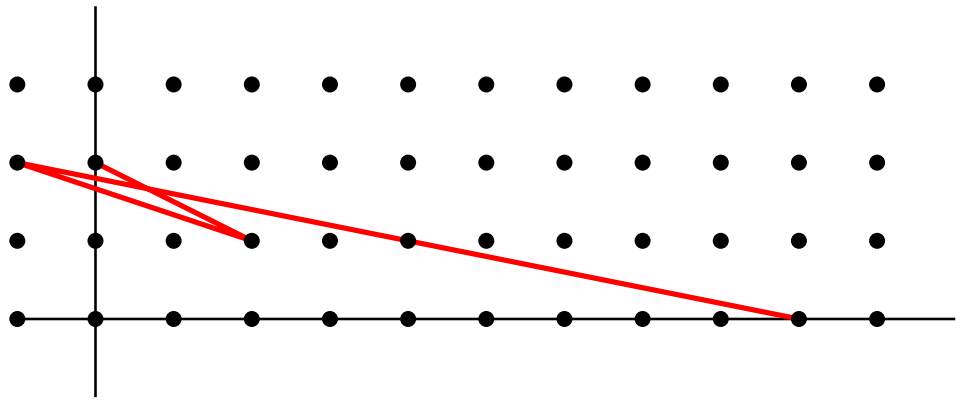
$$\mu(f,0) = (-1)^{n+1} + \sum_F (-1)^{n-s} s!\,\mathrm{Vol}_s(F) m_F = \nu(\Gamma_-(f)).$$

Example, the E_8 singularity

Example

The E_8 singularity $f(x,y,z) = x^2 + y^3 + z^5$ is Newton nondegenerate.





$$\mathrm{AJ}(f,0) = \left\{\frac{30}{6}\right\} - \left\{\frac{6}{2}\right\} - \left\{\frac{10}{2}\right\} - \left\{\frac{15}{3}\right\}$$
$$+ \left\{\frac{2}{1}\right\} + \left\{\frac{3}{1}\right\} + \left\{\frac{5}{1}\right\} = 2\left\{\frac{5}{1}\right\} - \left\{\frac{3}{1}\right\} + \left\{\frac{2}{1}\right\}$$

$$\mathcal{L}(f,0) = \deg\left(\mathrm{AJ}(f,0)\right) = 5/1 - 1 = 4$$

The BKO conjecture

Conjecture (Brzostowski, Krasiński and Oleksik '12)

If f is Newton nondegenerate, then the Newton diagram determines the Łojasiewicz exponent

$$\mathcal{L}(f,0) = \max\left\{ \frac{m_F}{n_F} - 1 \,\middle|\, F \subset \Gamma(f) \text{ } \textit{nonexceptional} \text{ } \textit{facet} \right\}$$

Theorem (Lenarcik '98)

The conjecture holds for plane curves, $n = 1$.

Theorem (Brzostowski, Krasiński and Oleksik '12)

The conjecture holds for surfaces, $n = 2$.

Theorem (-)

The conjecture does not hold for $n = 3$.

The Łojasiewicz exponent and Newton diagrams

Theorem (-)

If f is Newon nondegenerate, not a Morse point with n odd, then there exists a coordinate facet $F \subset \Gamma(f)$ such that

$$\mathcal{L}(f,0) = \frac{m_F}{n_F} - 1.$$

In fact,

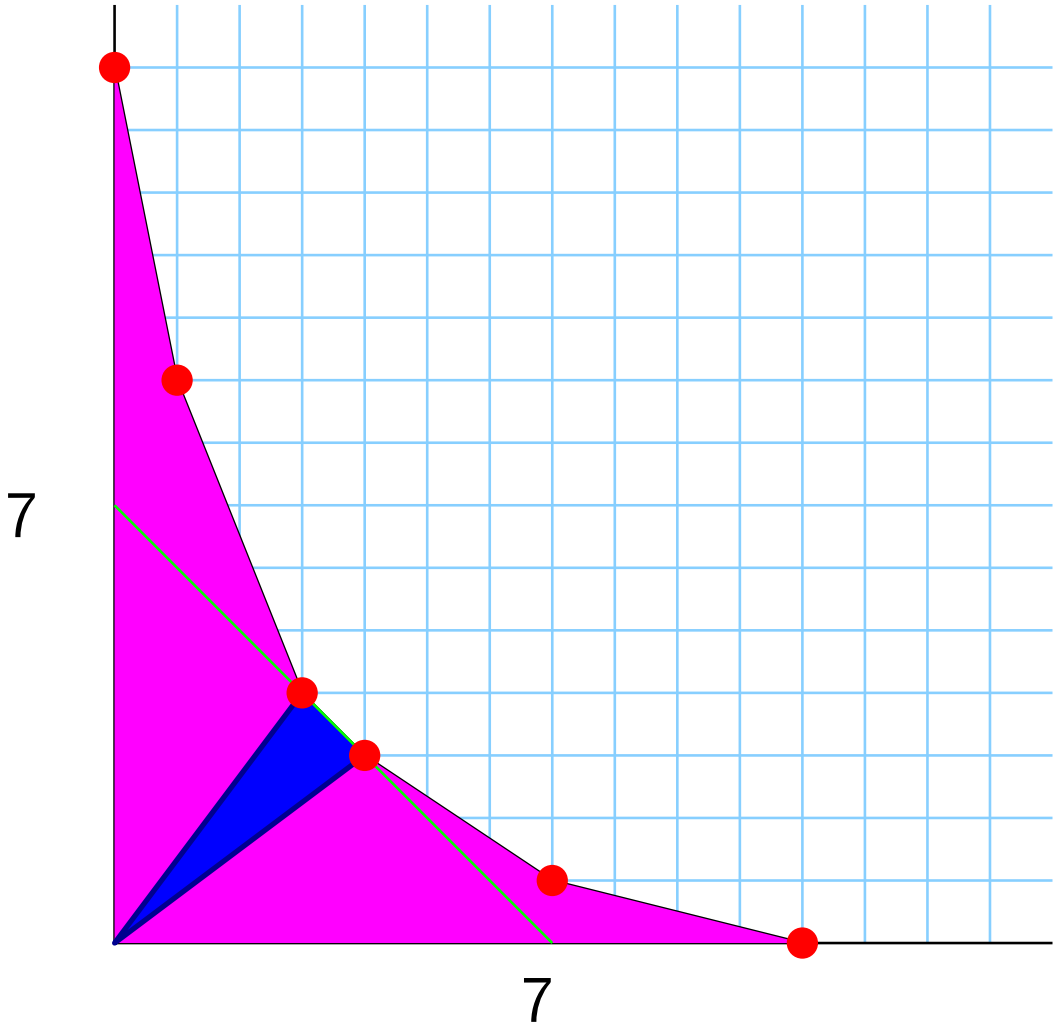
$$\sum \left\{ (-1)^{n-s} s! \operatorname{Vol}_s(F) m_F \;\middle|\; \frac{m_F}{n_F} > \alpha \right\} \geq 0 \tag{1}$$

for $\alpha \in \mathbb{Q}_{>0}$, with equality if and only if $\alpha - 1 \geq \mathcal{L}(f,0)$.

Remark

- *If $\alpha < \operatorname{mult}(f,0)$, then (1) is the Newton number of $\Gamma_-(f)$ (-1).*
- *In general, it is the Newton number of the complement of a cone.*

Example - level 7



Newton number( )

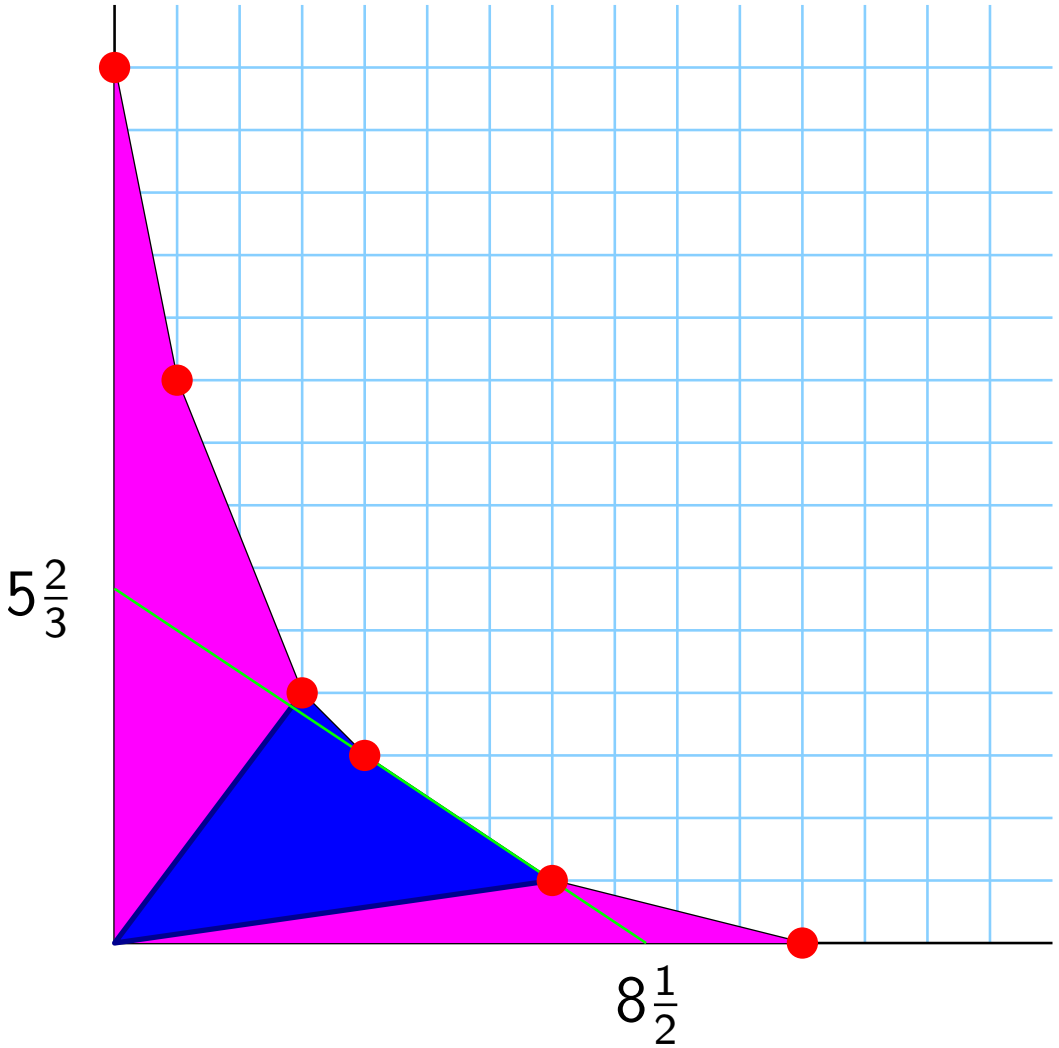
$$7 - 0 + 1 = 8$$

Newton number( )

$$65 - 11 - 14 = 40$$

$$\mathcal{L}(f, 0) > 7 - 1 = 6$$

Example - level $8\frac{1}{2}$



Newton number( )

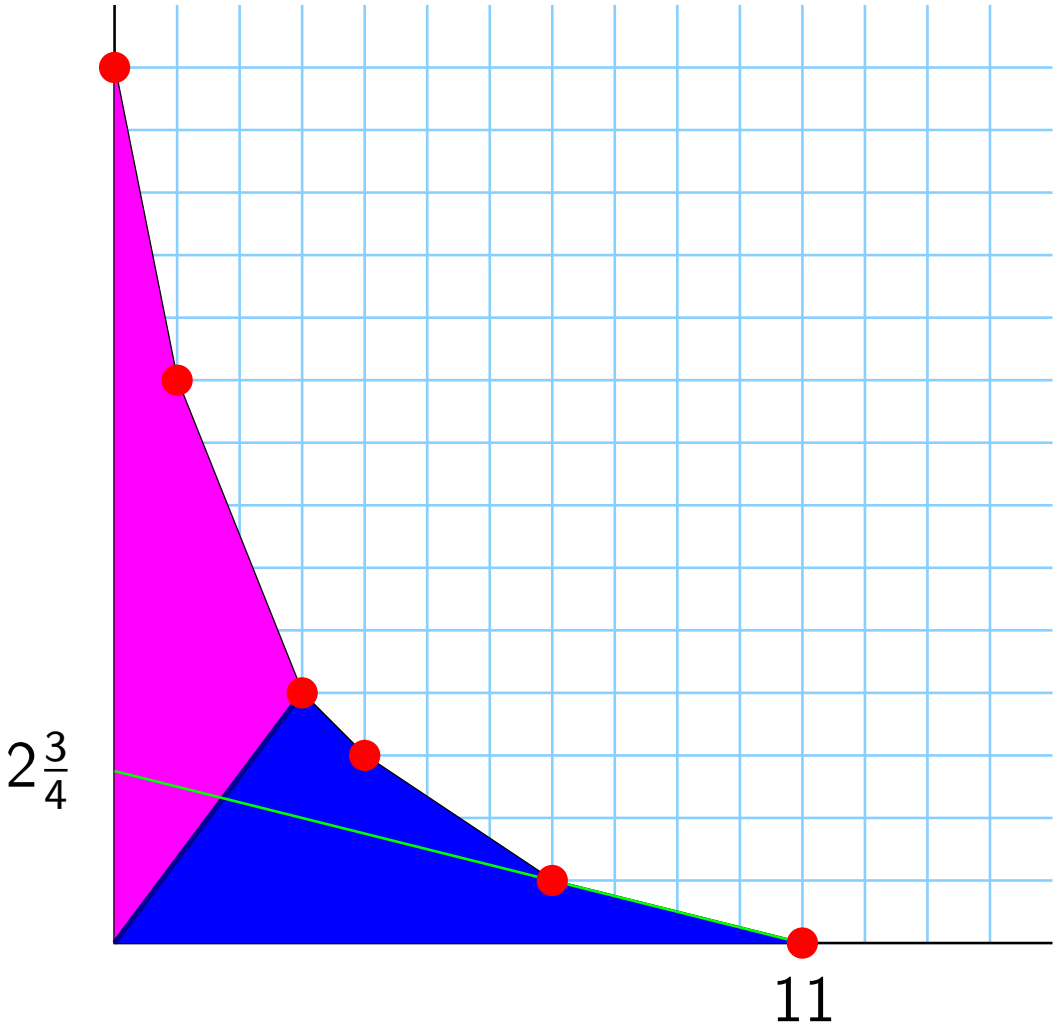
$$24 - 0 + 1 = 25$$

Newton number( )

$$48 - 11 - 14 = 23$$

$$\mathcal{L}(f, 0) > 8\tfrac{1}{2} - 1 = 7\tfrac{1}{2}$$

Example - level 11



Newton number( )

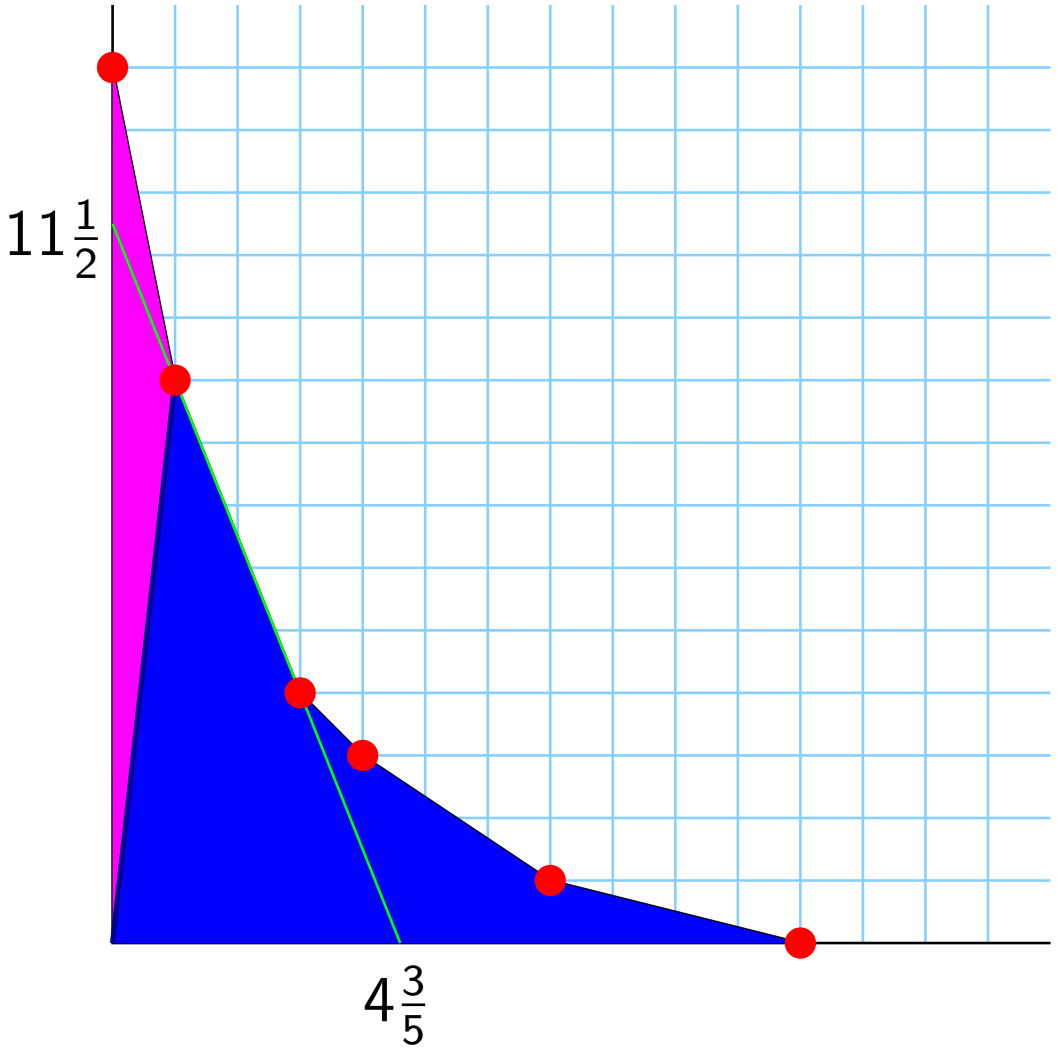
$$35 - 11 + 1 = 25$$

Newton number( )

$$37 - 14 = 23$$

$$\mathcal{L}(f, 0) > 11 - 1 = 10$$

Example - level $11\frac{1}{2}$



Newton number( )

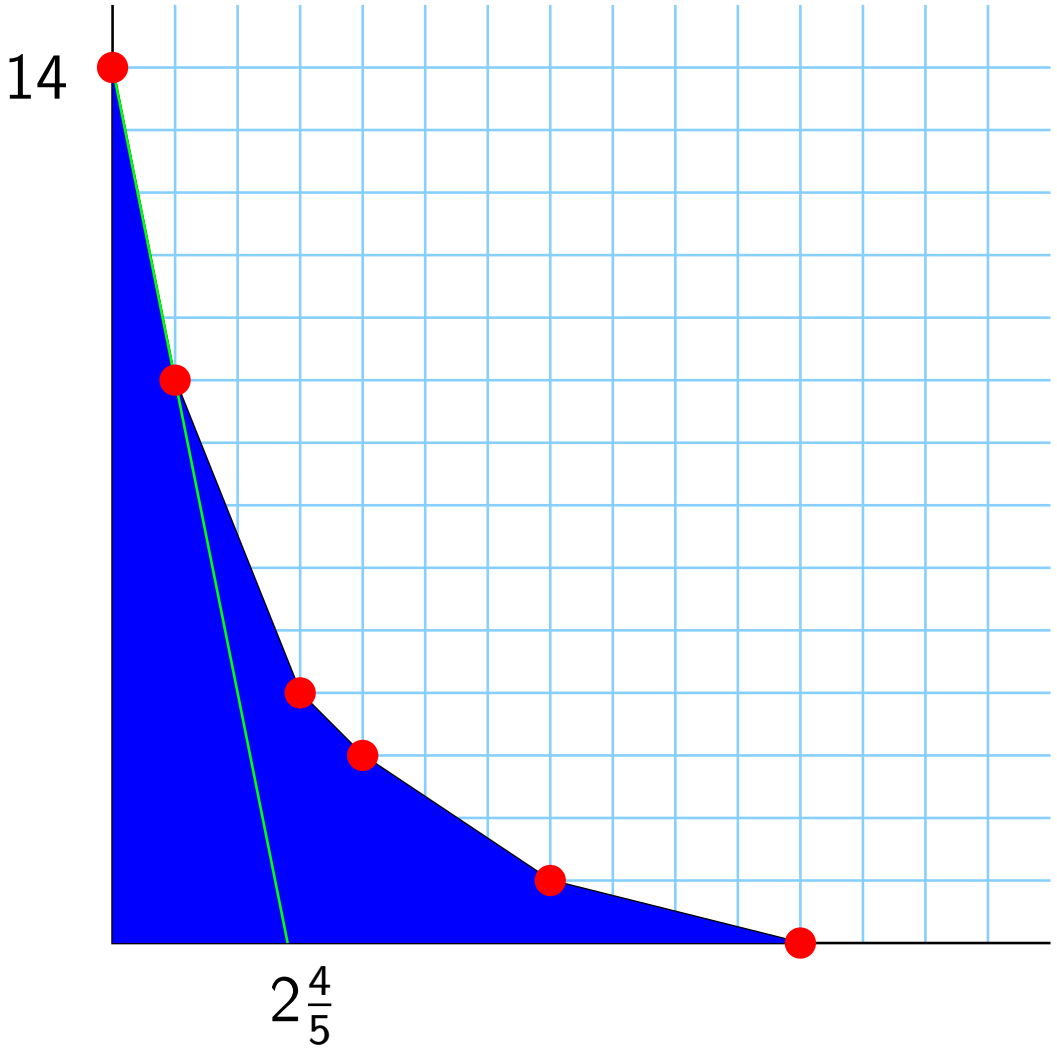
$$58 - 11 + 1 = 48$$

Newton number( )

$$11 - 11 = 0$$

$$\mathcal{L}(f, 0) = 11\tfrac{1}{2} - 1 = 10\tfrac{1}{2}$$

Example - level 14



Newton number( ) $= 72 - 11 - 14 + 1 = 48$

Newton number( ) $= 0$

Exceptional facets

- The previous theorem suggests defining *exceptional* facets as ones which do not contribute to the Newton number.

«Taux d'évanescence» on the Morse–Smale complex

The inequality (1) is the result of a technical result, which we suspect will have other interesting consequences.

- Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an isolated singularity.
- Let $F = F^{(n+1)}$ be its Milnor fiber.
- Let $F^{(d+1)} = F \cap H^{(d+1)}$, where $H^{(d+1)} = \{x_{d+1} = \cdots x_n = 0\}$.
- The pair $(F^{(d+1)}, F^{(d)})$ is a relative handlebody, with

$$\mu^{(d+1)} + \mu^{(d)}$$

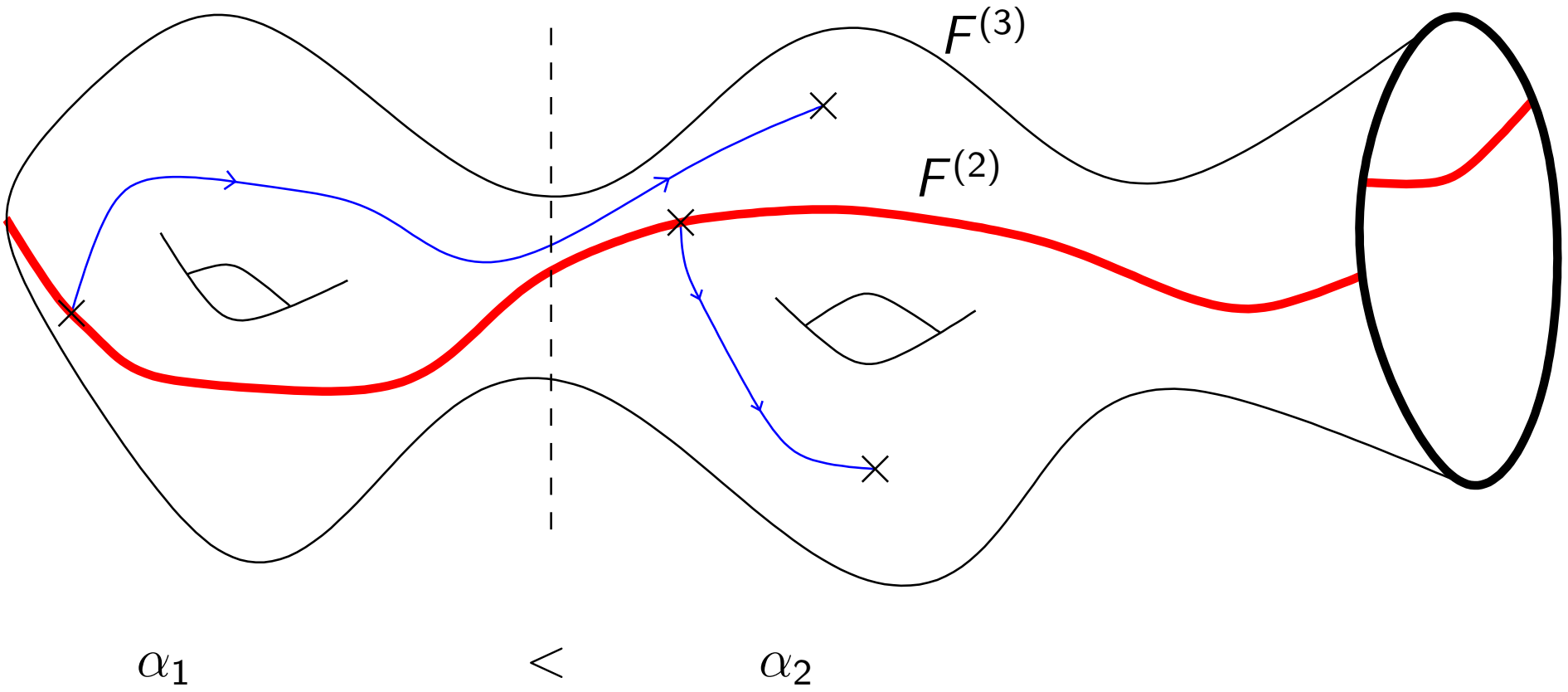
handles of index d corresponding to intersection points $F^{(d+1)} \cap P^{(d+1)}$ with the polar curve of the restriction of f to $H^{(d+1)}$.

- Each branch P_q of the polar curve has a *vanishing rate* $\alpha_q = m_q/n_q$.
- The Milnor fiber is a handle body, each handle has a vanishing rate.

«Taux d'évanescence» on the Morse–Smale complex

Theorem (-)

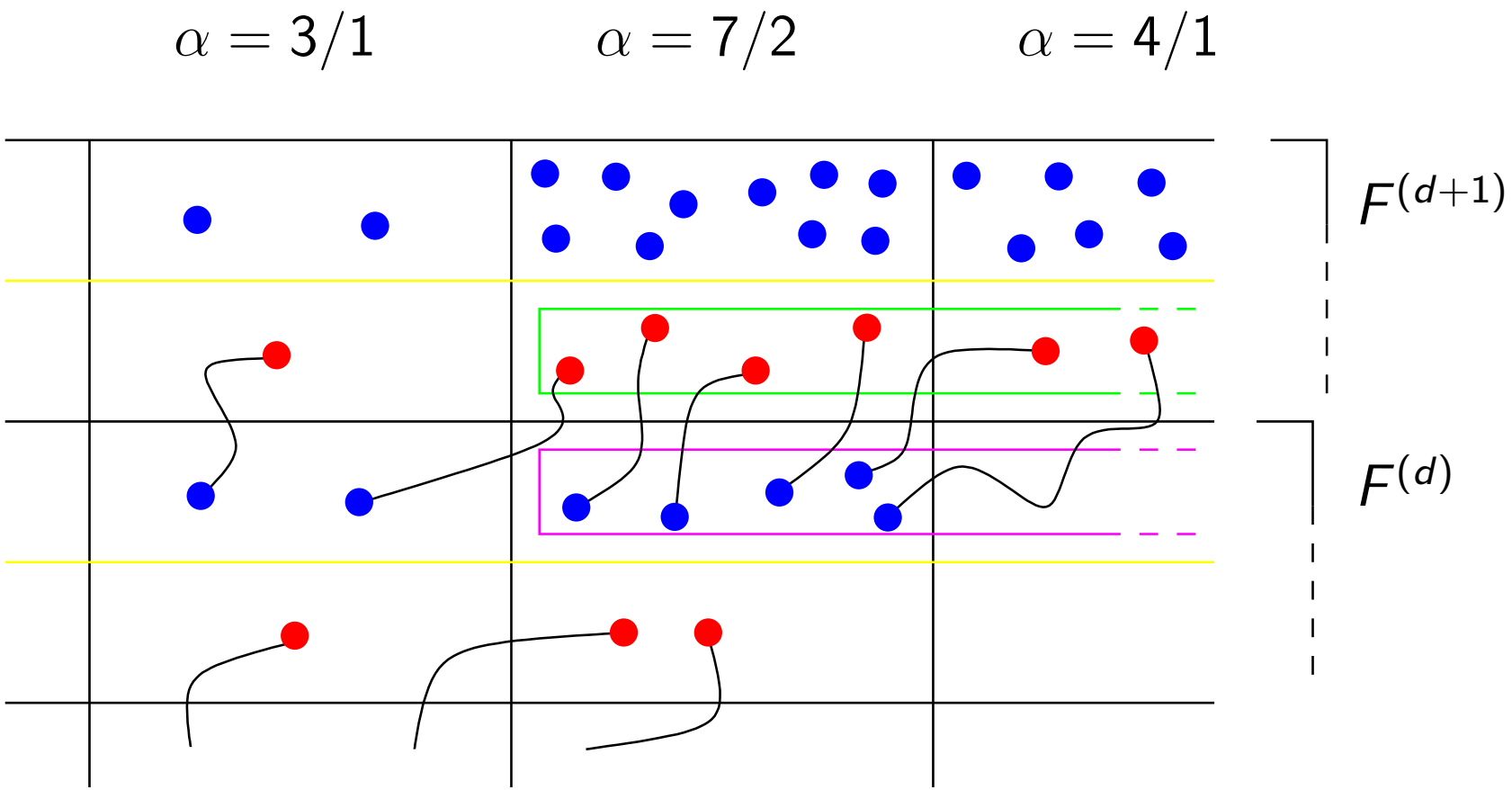
The increasing filtration by vanishing rates is a filtration on the associated Morse–Smale complex.



Generators of the Morse–Smale complex

- The Morse–Smale complex is generated in degree d by the intersection points
$$P^{(d+1)} \cap F^{(d+1)}$$
- Thus, we have $m_q^{(d+1)}$ generators for each branch $P_q^{(d+1)}$.
- Out of these, we can choose $n_q^{(d+1)}$ whose images form a basis for the image of the differential.
- We have
$$\sum_{q=1}^{l^{(d+1)}} m_q = \mu^{(d+1)} + \mu^{(d)}, \qquad \sum_{q=1}^{l^{(d+1)}} n_q = \mu^{(d)}.$$

Schematic of the Morse–Smale complex



A nested sequence of Milnor fibers in generic linear subspaces

$$F^{(1)} \subset F^{(2)} \subset \ldots \subset F^{(n)} \subset F^{(n+1)}$$

Some consequences

- Choose any $\alpha \leq \deg(J(f,0))$.
- Denote by a_d the number of critical points of index d with vanishing rate $\geq \alpha$.
- By counting the points, we find

$$a_n \geq (\alpha - 1)a_{d-1} \geq \ldots \geq (\alpha - 1)^{(d-1)}a_1.$$

- Since $a_n \neq 0$, we have a non-zero alternating sum

$$\sum_{d=0}^{n}(-1)^{n-d}a_d \neq 0.$$

- It follows that $\deg(AJ(f,0)) \geq \alpha$, and so (modulo Morse points):

$$\deg(AJ(f,0)) \geq \deg(J(f,0)).$$

Takk kærlega!

Dziękuję bardzo!

¡Muchas gracias!