Łojasiewiecz inequalities and applications to geometric analysis and gauge theory

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Monday, September 21, 2026

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Outline

1. Historical origins
2. Resolution of singularities and the Łojasiewicz inequality
3. Łojasiewicz–Simon gradient inequalities on Banach spaces
4. Morse–Bott property of functions with Łojasiewicz exponent 1/2
5. Survey of applications of the Łojasiewicz gradient inequality
6. Bibliography

References I

This talk is based in part on our related publications below:

- *Energy gap for Yang–Mills connections, II: Arbitrary closed Riemannian manifolds*, Advances in Mathematics **312** (2017), 547–587.
- *Optimal Łojasiewicz–Simon inequalities and Morse–Bott Yang–Mills energy functions*, Advances in Calculations of Variations 15 (2022), no. 4, 635-671.
- Resolution of singularities and geometric proofs of the Lojasiewicz inequalities, Geometry & Topology 23 (2019), no. 7, 3273-3313.
- (with M. Maridakis), *Łojasiewicz–Simon gradient inequalities for analytic and Morse–Bott functions on Banach spaces and applications to harmonic maps*, Journal für die Reine und Angewändte Mathematik 765 (2020), pp. 35-67.

References II

- (with M. Maridakis), *Łojasiewicz–Simon gradient inequalities for coupled Yang–Mills energy functions*, Memoirs of the American Mathematical Society (2020), volume 267, number 1302, xvi+138 pages.
- *On the Morse–Bott property of analytic functions on Banach spaces with Łojasiewicz exponent one half*, Calculus of Variations and Partial Differential Equations 59 (2020), no. 2, 50 pages.
- *Global existence and convergence of solutions to gradient systems and applications to Yang–Mills gradient flow*, arXiv:1409.1525.
- (with M. Maridakis) *Łojasiewicz–Simon gradient inequalities for the harmonic map energy function*, 33 pages, arXiv:1903.01953.

If you would like a copy of this presentation, please contact me at feehan@math.rutgers.edu.

Historical origins

Historical origins I

Laurent Schwartz [76] posed a fundamental division problem for distributions:

If f is a nonzero real-analytic function on an open set $\Omega \subset \mathbb{R}^n$, is multiplication by f surjective on the space of distributions $\mathscr{D}'(\Omega)$?

Equivalently, given $T \in \mathscr{D}'(\Omega)$, does there always exist $S \in \mathscr{D}'(\Omega)$ with

$$f\,S = T?$$

(When f is a polynomial, a related question arises when considering existence of fundamental solutions of constant coefficient partial dfferential equations — see, for example, Malgrange [62].)

Historical origins II

In [76], Schwartz had solved the division problem for complex manifolds, that is, if H is a holomorphic function on the complex manifold V and T is a current on V, then there is always a current S on V such that $HS = T$.

The answer is affirmative in the real analytic case too.

Indeed, Stanisław Łojasiewicz proved the general real analytic result in work announced in [53] and published in [55, 56], using ideas that became fundamental in the theory of semianalytic sets, stratification, and what we now call *Łojasiewicz inequalities*, which Łojasiewicz developed in [58, 59].

Lars Hörmander had independently treated the polynomial case of the division problem in [45].

In [5], Michael Atiyah gave a strikingly different proof of the division conjecture based on Hironaka's resolution of singularities [43].

Historical origins III

(Atiyah's article [5] has over 105 PDE-related citations, but only 2 Łojasiewicz-related citations, Biviá and Ausina [12] and F [28].)

Resolution of singularities replaces an arbitrary analytic zero set by a normal-crossing divisor, reducing the analytic problem to the meromorphic continuation of elementary monomial distributions.

Atiyah's method of proof of the division conjecture via resolution of singularities can readily be adapted to prove the Łojasiewicz inequalities.

Bierstone and Milman [10] derive the Łojasiewicz inequalities via resolution of singularities, a method rediscovered by us in F [28].

Modern approaches to and generalizations of the Łojasiewicz inequalities, such as the *Kurdyka–Łojasiewicz inequality*, use *o-minimal structures* – see Kurdyka [51], as well as Bolte, Daniilidis, Lewis, and Shiota [13], van den Dries and Miller [91], and van den Dries [90].

Historical origins IV

An important application of Łojasiewicz's *gradient* inequality (and infinite-dimensional generalizations thereof) arises in the study of non-linear evolution equations, a powerful idea developed by Leon Simon in

> *"Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems"*, *Ann. Math.* **118** *(1983), 525–571*

Simon's idea is based in part on a paradigm due to Łojasiewicz [60] for proving global existence and convergence of solutions to ordinary differential equations in Euclidean space.

Historical origins V

Theorem 1.1 (Global existence and convergence of solutions to gradient systems in $\mathbb{R}^n$)

(Łojasiewicz [60, Theorem 1], [61, p. 1592]) Let $\mathscr{E}$ be an analytic, non-negative function on a neighborhood of the origin in $\mathbb{R}^n$ such that $\mathscr{E}(0) = 0$. Then there exists a neighborhood, $U = \{x \in \mathbb{R}^n : |x| < \sigma\}$, of the origin such that each trajectory, $u_{x_0}(t)$, with $u_{x_0}(0) = x_0 \in U$, of the system,

$$\frac{\partial u}{\partial t}(t) = -\mathscr{E}'(u(t)), \tag{1}$$

is defined on $[0, \infty)$, has finite length, and converges uniformly to a point $z_\infty \in \operatorname{Crit} \mathscr{E} := \{z \in U : \mathscr{E}'(z) = 0\}$ as $t \to \infty$.

To prove Theorem 1.1, Łojasiewicz applied the following version of his gradient inequality [59]:

Historical origins VI

Theorem 1.2 (Finite-dimensional Łojasiewicz gradient inequality)

(Łojasiewicz [59]) Let $U \subset \mathbb{R}^n$ be an open subset, $z \in U$, and let $\mathscr{E} : U \to \mathbb{R}$ be a real-valued function. If $\mathscr{E}$ is real analytic on a neighborhood of z and $\mathscr{E}'(z) = 0$, then there exist constants $\theta \in (0, 1)$ and $\sigma > 0$ such that

$$\|\mathscr{E}'(x)\| \geq |\mathscr{E}(x) - \mathscr{E}(z)|^{\theta}, \quad \forall\, x \in \mathbb{R}^n,\ |x - z| < \sigma. \tag{2}$$

Resolution of singularities and geometric proofs of the Łojasiewicz inequalities

Łojasiewicz inequalities

Łojasiewicz inequalities I

The proof of the gradient inequality due to Łojasiewicz for a (real) analytic function on a neighborhood of a critical point in Euclidean space $\mathbb{R}^d$ appeared in

"Ensembles semi-analytiques", IHES preprint, 1965.

Łojasiewicz relied on his theory of semianalytic (and subanalytic) sets to prove his gradient inequality for functions on Euclidean space.

Łojasiewicz inequalities II

Gradient inequality using resolution of singularities

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, and $d \geq 1$ be an integer, and let $\mathbb{K}^{d*} = (\mathbb{K}^d)^*$ denote the dual space.

Theorem 1 (Łojasiewicz gradient inequality for an analytic function)

(See Łojasiewicz [59, Proposition 1, p. 92 (67)].) Let $d \geq 1$ be an integer, $U \subset \mathbb{K}^d$ be an open subset, and $\mathscr{E} : U \to \mathbb{K}$ be an analytic function. If $x_\infty \in U$ is a point such that $\mathscr{E}'(x_\infty) = 0$, then there are constants $C_0 \in (0, \infty)$, and $\sigma_0 \in (0, 1]$, and $\theta \in [1/2, 1)$ such that the differential map, $\mathscr{E}' : U \to \mathbb{K}^{d*}$, obeys

$$\|\mathscr{E}'(x)\|_{\mathbb{K}^{d*}} \geq C_0 |\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta, \quad \forall\, x \in B_{\sigma_0}(x_\infty), \tag{3}$$

where $B_{\sigma_0}(x_\infty) := \{x \in \mathbb{K}^d : \|x - x_\infty\|_{\mathbb{K}^d} < \sigma_0\} \subset U$.

Theorem 1 was stated by Łojasiewicz in [57, Theorem 4] and proved by him as [59, Proposition 1, p. 92]; see also Łojasiewicz [61, p. 1592].

Łojasiewicz inequalities III

An *infinite-dimensional* version of the gradient inequality (3) in Theorem 1 is proved by Simon as [78, Theorem 3, Equation (2.2)].

In [57], Łojasiewicz used his gradient inequality (Theorem 1) to give a positive answer to a question of Whitney:

> *If $\mathscr{F}$ is a real analytic function on an open set $U \subset \mathbb{R}^d$, then $\mathscr{F}^{-1}(0)$ is a deformation retract of its neighborhood.*

This deformation retract can be constructed using gradient flow.

Bierstone and Milman provided simplified proofs of Theorem 1 as [9, Proposition 6.8] and [10, Theorem 2.7].

Łojasiewicz inequalities IV

Our more direct proof of Theorem 1 makes it clear that one always has $\theta \geq 1/2$ and explain its geometric meaning, whereas previous proofs only assert the existence of $\theta \in (0, 1)$.

For applications to Geometric Analysis and Topology, it is essential to have $\theta < 1$, with $\theta = 1/2$ being the optimal exponent, corresponding to *exponential convergence* for the negative gradient flow defined by $\mathscr{E}$.

Łojasiewicz inequalities V

Gradient inequality without using resolution of singularities

Our proof of Theorem 1 in full generality employs *embedded resolution of singularities* (partly following Bierstone and Milman [10, Section 2]), but there are several weaker gradient inequalities that can be proved by far more elementary methods and those provide insight to applications in geometric analysis.

For example, when the function $\mathscr{E}$ in Theorem 1 is C^2 and Morse–Bott (a generalization of the concept of a Morse function that we will explain later), rather than an arbitrary real analytic function, one obtains the Łojasiewicz gradient inequality with exponent $\theta = 1/2$ as a consequence of the Mean Value Theorem (or Morse–Bott Lemma): see the forthcoming Theorem 2.4.

Łojasiewicz inequalities VI

As a stepping stone towards Theorem 1, we shall first establish a special case that holds for a class of C^1 functions.

Definition 2.1 (Function with simple normal crossings)

A function $f : U \to \mathbb{K}$ on an open neighborhood of the origin in $\mathbb{K}^d$ has *simple normal crossings* if

$$f(x) = x_1^{n_1} \cdots x_d^{n_d} f_0(x), \quad \forall\, x = (x_1, \ldots, x_d) \in U, \tag{4}$$

where $n_i \in \mathbb{Z} \cap [0, \infty)$ and f_0 is a function such that $f_0(0) \neq 0$.

Łojasiewicz inequalities VII

Theorem 2 (Łojasiewicz gradient inequality for a C^1 function with simple normal crossings and characterization of the optimal exponent and Morse–Bott condition)

Let $d \geq 2$ be an integer, $U \subset \mathbb{K}^d$ be an open neighborhood of the origin, and $\mathscr{E} : U \to \mathbb{K}$ be a C^1 function with simple normal crossings. If $\mathscr{E}'(0) = 0$, then there are constants $C_0 \in (0, \infty)$ and $\sigma \in (0, 1]$ such that

$$\|\mathscr{E}'(x)\|_{\mathbb{K}^{d*}} \geq C_0 |\mathscr{E}(x)|^\theta, \quad \forall\, x \in B_\sigma, \tag{5}$$

where $\theta = 1 - 1/N \in [1/2, 1)$ and $N = \sum_{i=1}^{c} n_i$ is the total degree of the monomial in the expression (4) for $\mathscr{E}$, with $c \leq d$ being the number of exponents $n_i \geq 1$ and obeying $c \geq 2$ or $c = 1$ and $n_1 \geq 2$.

Łojasiewicz inequalities VIII

Theorem 2 (Łojasiewicz gradient inequality for a C^1 function with simple normal crossings and characterization of the optimal exponent and Morse–Bott condition)

In particular, $\theta = 1/2$ if and only if $c = 2$ and (after relabeling coordinates), $n_1 = n_2 = 1$ or $c_1 = 1$ and $n_1 = 2$, and furthermore, if $\mathscr{E}$ is C^2, then it is Morse–Bott on B_σ.

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Remark 2.2 (Geometry of the critical set)

Theorem 2.4 shows that, when $\mathscr{E}$ is Morse–Bott and so its critical set is a smooth submanifold, then its Łojasiewicz exponent θ is equal to $1/2$.
Conversely, when $\theta = 1/2$, Theorem 2 implies that (after relabeling)

$$f(x) = x_1^2 f_0(x) \quad \text{or} \quad f(x) = x_1 x_2 f_0(x),$$

so $B_\sigma \cap \operatorname{Crit} \mathscr{E} = \{x_1 = 0\} \cap B_\sigma$ or $\{x_1 = x_2 = 0\} \cap B_\sigma$, a codimension one or two smooth submanifold of B_σ.
Theorem 1 is proved by applying resolution of singularities to an ideal defined by an arbitrary real analytic function $\mathscr{E}$ and applying Theorem 2 to the resulting monomial (the product of $x_1^{n_1} \cdots x_d^{n_c}$ and a non-vanishing analytic function).

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Remark 2.2 (Geometry of the critical set)

Consequently, if $\theta = 1/2$ then there is a constraint on the nature of the singularities in the critical set of $\mathscr{E}$.
Our proof of Theorem 1 shows that application of resolution of singularities does not change the Łojasiewicz exponent and so it is an interesting question whether one can characterize the class of real analytic functions with $\theta = 1/2$.
We shall return to this question later in our talk.

For comparison, we recall the

Łojasiewicz inequalities XI

Definition 2.3 (Morse–Bott function)

Let $d \geq 1$ be an integer, $U \subset \mathbb{K}^d$ be an open subset, $\mathscr{E} : U \to \mathbb{K}$ be a C^2 function, and $\operatorname{Crit} \mathscr{E} := \{x \in U : \mathscr{E}'(x) = 0\}$. We say that $\mathscr{E}$ is *Morse–Bott at a point* $x_\infty \in \operatorname{Crit} \mathscr{E}$ if

1. $\operatorname{Crit} \mathscr{E}$ is a C^2 submanifold of U, and
2. $T_{x_\infty} \operatorname{Crit} \mathscr{E} = \operatorname{Ker} \mathscr{E}''(x_\infty)$ when $\mathscr{E}''(x_\infty)$ is considered as an operator in $\operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^d, \mathbb{K}^{d*})$,

where $T_x \operatorname{Crit} \mathscr{E}$ is the tangent space to $\operatorname{Crit} \mathscr{E}$ at a point $x \in \operatorname{Crit} \mathscr{E}$.

Łojasiewicz inequalities XII

Theorem 2.4 (Łojasiewicz gradient inequality for a Morse–Bott function on Euclidean space)

Let $d \geq 1$ be an integer and $U \subset \mathbb{K}^d$ an open subset. If $\mathscr{E} : U \to \mathbb{K}$ is a Morse–Bott function, then there are constants $C_0 \in (0, \infty)$ and $\sigma_0 \in (0, 1]$ such that

$$\|\mathscr{E}'(x)\|_{\mathbb{K}^{d*}} \geq C_0 |\mathscr{E}(x) - \mathscr{E}(x_\infty)|^{1/2}, \quad \forall\, x \in B_{\sigma_0}(x_\infty). \tag{6}$$

Łojasiewicz inequalities XIII

Consequences of the gradient inequality

The gradient inequality (3) yields useful corollaries.

Corollary 3 (Łojasiewicz inequality for distance to critical set)

Let $d \geq 1$ be an integer, $U \subset \mathbb{K}^d$ be an open subset, and $\mathscr{E} : U \to \mathbb{K}$ be C^1. If $x_\infty \in \operatorname{Crit} \mathscr{E}$ and the gradient inequality (3) holds with exponent θ for $\mathscr{E}$ near x_∞, then there are constants $C_1 \in (0, \infty)$, and $\sigma_1 \in (0, 1]$, and $\beta = 1/(1-\theta) \in [2, \infty)$ such that

$$|\mathscr{E}(x)| \geq C_1 \operatorname{dist}(x, \operatorname{Crit} \mathscr{E})^\beta, \quad \forall\, x \in B_{\sigma_1}(x_\infty), \tag{7}$$

where $\operatorname{dist}(x, \operatorname{Crit} \mathscr{E}) := \inf\{\|x - c\|_{\mathbb{K}^d} : c \in \operatorname{Crit} \mathscr{E}\}$.

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Corollary 4 (Łojasiewicz inequality for distance to zero set)

(See Łojasiewicz [59, Theorem 2, p. 85 (62)].) Let $d \geq 1$ be an integer, $U \subset \mathbb{K}^d$ be an open subset, and $\mathscr{E} : U \to \mathbb{K}$ be C^1. If $x_0 \in Z := \mathscr{E}^{-1}(0)$ and the gradient inequality (3) holds with exponent θ for the function $\mathscr{E}^2$ near x_0, then there are constants $C_2 \in (0, \infty)$, and $\sigma_2 \in (0, 1]$, and $\alpha = 1/(2(1-\theta)) \in [1, \infty)$ such that

$$|\mathscr{E}(x)| \geq C_2 \operatorname{dist}(x, Z)^\alpha, \quad \forall\, x \in B_{\sigma_2}(x_0), \tag{8}$$

where $\operatorname{dist}(x, Z) := \inf\{\|x - z\|_{\mathbb{K}^d} : z \in Z\}$.

Corollary 4 was stated by Łojasiewicz in [57, Corollary, p. 88] and proved by him in [55, 56] and in [59] (for $\mathbb{K} = \mathbb{R}$), with simplified proofs provided by Bierstone and Milman as [9, Theorem 6.4 and Remark 6.5] (for $\mathbb{K} = \mathbb{R}$) and [10, Theorem 2.8] (for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

When $\mathscr{E}$ is a real polynomial on $\mathbb{R}^d$, Corollary 4 is due to Hörmander [45, Lemma 1].

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Combining Theorem 1 and Corollary 3 gives (with $\gamma = \beta\theta$)

Corollary 5 (Łojasiewicz gradient-distance inequality)

Let $d \geq 1$ be an integer, $U \subset \mathbb{K}^d$ be an open neighborhood of a point x_∞, and $\mathscr{E} : U \to \mathbb{K}$ be a C^1 function. If $\mathscr{E}'(x_\infty) = 0$ and the gradient inequality (3) holds for $\mathscr{E}$ near x_∞, then there are constants $C_3 \in (0, \infty)$, and $\sigma_3 \in (0, 1]$, and $\gamma \in [1, \infty)$ such that the differential map, $\mathscr{E}' : U \to \mathbb{K}^{d*}$, obeys

$$\|\mathscr{E}'(x)\|_{\mathbb{K}^{d*}} \geq C_3 \operatorname{dist}(x, \operatorname{Crit} \mathscr{E})^\gamma, \quad \forall\, x \in B_{\sigma_3}(x_\infty). \tag{9}$$

The inequality (9) is stated by Simon in [78, Equation (2.3)] (for $\mathbb{K} = \mathbb{R}$, though with $\gamma \in [2, \infty)$) and attributed by him to Łojasiewicz [59].

An *infinite-dimensional* version of the distance inequality (9) in Corollary 5 is proved by Simon as [78, Theorem 3, Equation (2.1)], using his [78, Theorem 3, Equation (2.2)].

Main ideas in proofs

Main ideas in proofs I

We prove Theorem 1 by appealing to resolution of singularities for analytic sets but in a different way from that of Bierstone and Milman [10] and that approach allows us to give the forthcoming partial identification (12) of the Łojasiewicz exponent, θ, and show that it is restricted to the interval $[1/2, 1)$, sharpening the range $(0, 1)$ provided in [9, 10, 59].

Resolution of singularities for analytic varieties yields the following special case of results due to Hironaka:

Main ideas in proofs II

Theorem 2.5 (Monomialization of an analytic function)

Let $U \subset \mathbb{K}^d$ be an open neighborhood of the origin and $f : U \to \mathbb{K}$ be an analytic function. If $f(0) = 0$ then, after possibly shrinking U, there are an open neighborhood $V \subset \mathbb{K}^d$ of the origin and an analytic map,

$$\varphi : V \ni y \mapsto x \in U, \tag{10}$$

such that $\varphi(0) = 0$ and π restricts to an analytic diffeomorphism on the complement of the zero set, $f^{-1}(0)$,

$$\varphi : V \setminus \varphi^{-1}(f^{-1}(0)) \cong U \setminus f^{-1}(0),$$

and $\varphi^* f$ is a simple normal crossing function, that is,

$$\varphi^* f(y) = y_1^{n_1} y_2^{n_2} \cdots y_d^{n_d}, \quad \forall\, y \in V, \tag{11}$$

where the n_i are non-negative integers for $i = 1, \ldots, d$.

Main ideas in proofs III

Remarks on resolution of singularities

A *resolution of singularities* of an algebraic or analytic variety X is a proper birational morphism, $\pi : \widetilde{X} \to X$, such that $\widetilde{X}$ is non-singular.

Hironaka [43] proved that any algebraic variety over any field of characteristic zero admits a resolution of singularities and, moreover, that both complex and real analytic varieties admit resolutions of singularities as well [3, 4, 44].

Bierstone and Milman [10] (see [11] for their expository introduction to [10]) have developed a proof of resolution of singularities that applies to real and complex analytic varieties and to algebraic varieties over any field of characteristic zero and which significantly shortens and simplifies Hironaka's proof.

Main ideas in proofs IV

Łojasiewicz exponent of a monomial function

The Łojasiewicz exponent of a monomial function can be computed exactly using the *Generalized Young Inequality* (13):

Lemma 2.6 (Łojasiewicz exponent of a monomial function)

(See F [28, Theorem 5] or Haraux [40, Theorem 3.1].) Let $g : \mathbb{K}^d \to \mathbb{K}$ be an analytic function given by $g(y) = y_1^{n_1} y_2^{n_2} \cdots y_d^{n_d}$ for $y \in \mathbb{K}^d$, where the n_i are non-negative integers for $i = 1, \ldots, d$. If $g(0) = 0$ and $g'(0) = 0$, then g obeys the Łojasiewicz gradient inequality (3) on $U = \mathbb{K}^d$, namely

$$\|g'(x)\|_{\mathbb{K}^{d*}} \geq C|g(x)|^\theta, \quad \forall\, x \in \mathbb{K}^d,$$

for a constant $C \in (0, \infty)$ and exponent

$$\theta = 1 - \frac{1}{N} \in [1/2, 1), \quad \text{where } N := \sum_{i=1}^{d} n_i \geq 2. \tag{12}$$

Main ideas in proofs V

For integers $c \geq 2$, the Generalized Young Inequality (see Hardy, Littlewood, and Pólya [42, Inequality (2.5.2)]) is

$$\left(\prod_{j=1}^{c} a_j\right)^r \leq r \sum_{j=1}^{c} \frac{a_j^{p_j}}{p_j}, \tag{13}$$

for constants $a_j > 0$ and $p_j > 0$ and $r > 0$ such that $\sum_{j=1}^{c} 1/p_j = 1/r$.

We apply Theorem 2.5 and Lemma 2.6 to prove Theorem 1 using the elementary

Main ideas in proofs VI

Lemma 2.7 (Łojasiewicz exponents and maps)

Let d, e be positive integers, $V \subset \mathbb{K}^e$ and $U \subset \mathbb{K}^d$ be open neighborhoods of the origins and $\phi : V \to U$ be an open C^1 map such that $\phi(0) = 0$. If $f : U \to \mathbb{K}$ is a C^1 function such that $\phi^* f$ obeys the Łojasiewicz gradient inequality (3) at the origin with exponent $\theta \geq 0$ then, after possibly shrinking U, the function f obeys the Łojasiewicz gradient inequality (3) with the same exponent θ and a possibly smaller constant $C \in (0, \infty)$.

Łojasiewicz–Simon inequalities for real analytic and Morse–Bott functions on Banach spaces

Łojasiewicz–Simon gradient inequalities on Banach spaces

Łojasiewicz–Simon gradient inequalities on Banach spaces

Łojasiewicz–Simon inequalities on Banach spaces I

We begin with the following generalization of Simon's infinite-dimensional version [78, Theorem 3] of the Łojasiewicz gradient inequality [59].

Theorem 3.1 (Łojasiewicz–Simon gradient inequality for analytic functions on Banach spaces)

(See F. and Maridakis [30, Theorem 1].) Let $\mathscr{X}$ be a Banach space that is continuously embedded in a Hilbert space $\mathscr{H}$. Let $\mathscr{U} \subset \mathscr{X}$ be an open subset, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ be an analytic function, and $x_\infty \in \mathscr{U}$ be a critical point of $\mathscr{E}$, that is, $\mathscr{E}'(x_\infty) = 0$. Assume that $\mathscr{E}''(x_\infty) : \mathscr{X} \to \mathscr{X}^*$ is a Fredholm operator with index zero. Then there are positive constants, Z, σ, and $\theta \in [1/2, 1)$, with the following significance. If $x \in \mathscr{U}$ obeys

$$\|x - x_\infty\|_{\mathscr{X}} < \sigma, \tag{14}$$

then

$$\|\mathscr{E}'(x)\|_{\mathscr{X}^*} \geq Z|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta. \tag{15}$$

Theorem 3.1 appears without proof in Huang [48, Theorem 2.4.5].

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Łojasiewicz–Simon inequalities on Banach spaces II

Simon [78, Theorem 3] generalized the Łojasiewicz gradient inequality for analytic functions on Euclidean space to certain analytic functions on the *Banach space of $C^{2,\alpha}$ sections of vector bundles over a closed, Riemannian manifold*.

Many researchers have adapted or extended Simon's [78, Theorem 3] in the intervening years, including Chill, Feireisl, Haraux, Huang, Jendoubi, Råde, Takáč, and others.

Our proof of Theorem 3.1 generalizes that of Feireisl and Takáč for their [34, Proposition 6.1] in the case of the Ginzburg-Landau energy function.

The [20, Theorem 3.10 and Corollary 3.11] and [21, Corollary 3] due to Chill provide alternative versions of the Łojasiewicz–Simon gradient inequality for an analytic function on a Banach space, but the hypotheses of Theorem 3.1 and the following extensions are simpler and more general.

Łojasiewicz–Simon inequalities on Banach spaces III

The hypothesis in Theorem 3.1 that $\mathscr{E}''(x_\infty) : \mathscr{X} \to \mathscr{X}^*$ is a Fredholm operator is restrictive in practice (for the Yang–Mills energy function over a manifold X of dimension $d \geq 5$ or for the harmonic map energy function, for example).

Definition 3.2 (Gradient map)

(See [8, Section 2.5], [48, Definition 2.1.1].) Let $\mathscr{U} \subset \mathscr{X}$ be an open subset of a Banach space, $\mathscr{X}$, and let $\tilde{\mathscr{X}}$ be a Banach space with continuous embedding, $\tilde{\mathscr{X}} \subseteq \mathscr{X}^*$. A continuous map, $\mathscr{M} : \mathscr{U} \to \tilde{\mathscr{X}}$, is called a *gradient map* if there exists a C^1 function, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$, such that

$$\mathscr{E}'(x)v = \langle v, \mathscr{M}(x)\rangle_{\mathscr{X}\times\mathscr{X}^*}, \quad \forall\, x \in \mathscr{U}, \quad v \in \mathscr{X}, \tag{16}$$

where $\langle \cdot, \cdot \rangle_{\mathscr{X}\times\mathscr{X}^*}$ is the canonical bilinear form on $\mathscr{X} \times \mathscr{X}^*$. The real-valued function, $\mathscr{E}$, is called a *potential* for the gradient map, $\mathscr{M}$.

Łojasiewicz–Simon inequalities on Banach spaces IV

Theorem 3.3 (Łojasiewicz–Simon gradient inequality for analytic functions on Banach spaces)

(See F. and Maridakis [30, Theorem 2].) Let $\mathscr{X}$ and $\tilde{\mathscr{X}}$ be Banach spaces with continuous embeddings, $\mathscr{X} \subset \tilde{\mathscr{X}} \subset \mathscr{X}^*$, and such that $\mathscr{X}$ is continuously embedded in a Hilbert space $\mathscr{H}$. Let $\mathscr{U} \subset \mathscr{X}$ be an open subset, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ be an analytic function, and $x_\infty \in \mathscr{U}$ be a critical point of $\mathscr{E}$, that is, $\mathscr{E}'(x_\infty) = 0$. Assume that $\mathscr{E}''(x_\infty) : \mathscr{X} \to \mathscr{X}^*$ is a Fredholm operator with index zero. Then there are constants $Z \in (0, \infty)$, and $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$, with the following significance. If $x \in \mathscr{U}$ obeys

$$\|x - x_\infty\|_{\mathscr{X}} < \sigma, \tag{17}$$

then

$$\|\mathscr{M}(x)\|_{\tilde{\mathscr{X}}} \geq Z|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta. \tag{18}$$

Theorem 3.3 reduces to Theorem 3.1 when $\tilde{\mathscr{X}} = \mathscr{X}^*$.

Łojasiewicz–Simon inequalities on Banach spaces V

While Theorem 3.3 has important applications to proofs of global existence, convergence, convergence rates, and stability of gradient flows defined by an energy function, $\mathscr{E} : \mathscr{X} \supset \mathscr{U} \to \mathbb{R}$, with gradient map, $\mathscr{M} : \mathscr{X} \supset \mathscr{U} \to \tilde{\mathscr{X}}$, the gradient inequality (18) is most useful when it has the form,

$$\|\mathscr{M}(x)\|_{\mathscr{H}} \geq Z|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta, \quad \forall\, x \in \mathscr{U} \text{ with } \|x - x_\infty\|_{\mathscr{X}} < \sigma,$$

where $\mathscr{H}$ is a Hilbert space and the Banach space, $\mathscr{X}$, is a dense subspace of $\mathscr{H}$ with continuous embedding, $\mathscr{X} \subset \mathscr{H}$, and so $\mathscr{H}^* \subset \mathscr{X}^*$ is also a continuous embedding.

Unfortunately, such a gradient inequality does not follow in useful generality from Theorem 3.3.

Łojasiewicz–Simon inequalities on Banach spaces VI

However, these gradient inequalities *are* implied by the forthcoming Theorem 3.4 which generalizes and simplifies Huang's [48, Theorem 2.4.2 (i)].

Theorem 3.4 (Generalized Łojasiewicz–Simon gradient inequality for analytic functions on Banach spaces)

(See F. and Maridakis [30, Theorem 3].) Let $\mathscr{X}$ and $\tilde{\mathscr{X}}$ be Banach spaces with continuous embeddings, $\mathscr{X} \subset \tilde{\mathscr{X}} \subset \mathscr{X}^*$, and such that $\mathscr{X}$ is continuously embedded in a Hilbert space $\mathscr{H}$. Let $\mathscr{U} \subset \mathscr{X}$ be an open subset, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ be an analytic function, and $x_\infty \in \mathscr{U}$ be a critical point of $\mathscr{E}$, that is, $\mathscr{E}'(x_\infty) = 0$. Let

$$\mathscr{X} \subset \mathscr{G} \subset \tilde{\mathscr{G}} \quad \text{and} \quad \tilde{\mathscr{X}} \subset \tilde{\mathscr{G}} \subset \mathscr{X}^*,$$

be continuous embeddings of Banach spaces such that the compositions,

$$\mathscr{X} \subset \mathscr{G} \subset \tilde{\mathscr{G}} \quad \text{and} \quad \mathscr{X} \subset \tilde{\mathscr{X}} \subset \tilde{\mathscr{G}},$$

induce the same embedding, $\mathscr{X} \subset \tilde{\mathscr{G}}$. Let $\mathscr{M} : \mathscr{U} \to \tilde{\mathscr{X}}$ be a gradient map for $\mathscr{E}$ in the sense of Definition 3.2.

Łojasiewicz–Simon inequalities on Banach spaces VII

Theorem 3.4 (Generalized Łojasiewicz–Simon gradient inequality for analytic functions on Banach spaces)

Suppose that for each $x \in \mathscr{U}$, the bounded, linear operator,

$$\mathscr{M}'(x) : \mathscr{X} \to \tilde{\mathscr{X}},$$

has an extension

$$\mathscr{M}_1(x) : \mathscr{G} \to \tilde{\mathscr{G}}$$

such that the map

$$\mathscr{U} \ni x \mapsto \mathscr{M}_1(x) \in \mathscr{L}(\mathscr{G}, \tilde{\mathscr{G}}) \quad \text{is continuous.}$$

If $\mathscr{M}'(x_\infty) : \mathscr{X} \to \tilde{\mathscr{X}}$ and $\mathscr{M}_1(x_\infty) : \mathscr{G} \to \tilde{\mathscr{G}}$ are Fredholm operators with index zero, then there are constants, $Z \in (0, \infty)$ and $\sigma \in (0, 1]$ and $\theta \in [1/2, 1)$, with the following significance.

Łojasiewicz–Simon inequalities on Banach spaces VIII

Theorem 3.4 (Generalized Łojasiewicz–Simon gradient inequality for analytic functions on Banach spaces)

If $x \in \mathscr{U}$ *obeys*

$$\|x - x_\infty\|_{\mathscr{X}} < \sigma, \tag{19}$$

then

$$\|\mathscr{M}(x)\|_{\tilde{\mathscr{G}}} \geq Z|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta. \tag{20}$$

Suppose now that $\tilde{\mathscr{G}} = \mathscr{H}$, a Hilbert space, so that

$$\mathscr{E}'(x)v = \langle v, \mathscr{M}(x)\rangle_{\mathscr{X}\times\mathscr{X}^*} = (v, \mathscr{M}(x))_{\mathscr{H}}, \quad \forall\, x \in \mathscr{U} \text{ and } v \in \mathscr{X},$$

using the continuous embeddings, $\tilde{\mathscr{X}} \subset \mathscr{H} \subset \mathscr{X}^*$.

Then by Theorem 3.4 if $x \in \mathscr{U}$ obeys

$$\|x - x_\infty\|_{\mathscr{X}} < \sigma, \tag{21}$$

Łojasiewicz–Simon inequalities on Banach spaces IX

then

$$\|\mathscr{M}(x)\|_{\mathscr{H}} \geq Z|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta, \tag{22}$$

as desired.

Łojasiewicz–Simon gradient inequalities for Morse–Bott functions on Banach spaces

Łojasiewicz–Simon inequalities for Morse–Bott functions I

It is of considerable interest to know when the optimal exponent $\theta = 1/2$ is achieved, since in that case one can prove (see [26, Theorem 24.21], for example) that a global solution, $u : [0, \infty) \to \mathscr{X}$, to a gradient system governed by the Łojasiewicz–Simon gradient inequality,

$$\frac{du}{dt} = -\mathscr{E}'(u(t)), \quad u(0) = u_0,$$

has exponential rather than mere power-law rate of convergence to the critical point, u_∞.

One simple version of such an optimal Łojasiewicz–Simon gradient inequality is provided in Huang [48, Proposition 2.7.1] which, although interesting, its hypotheses are very restrictive, a special case of Theorem 3.1 where $\mathscr{X}$ is a Hilbert space and the Hessian, $\mathscr{E}''(x_\infty) : \mathscr{X} \to \mathscr{X}^*$, is an invertible operator.

Łojasiewicz–Simon inequalities for Morse–Bott functions II

For the harmonic map energy function, a more interesting optimal Łojasiewicz–Simon-type gradient inequality,

$$\|\mathscr{E}'(f)\|_{L^p(S^2)} \geq Z|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^{1/2},$$

has been obtained by Kwon [52, Theorem 4.2] for maps $f : S^2 \to N$, where N is a closed Riemannian manifold and f is close to a harmonic map f_∞ in the sense that

$$\|f - f_\infty\|_{W^{2,p}(S^2)} < \sigma,$$

where p is restricted to the range $1 < p \leq 2$, and f_∞ is assumed to be *integrable* in the sense of [52, Definitions 4.3 or 4.4 and Proposition 4.1].

The [54, Lemma 3.3] due to Liu and Yang is another example of an optimal Łojasiewicz–Simon-type gradient inequality for the harmonic map energy function, but restricted to the setting of maps $f : S^2 \to N$, where

Łojasiewicz–Simon inequalities for Morse–Bott functions III

N is a Kähler manifold of complex dimension $n \geq 1$ and nonnegative bisectional curvature, and the energy $\mathscr{E}(f)$ is sufficiently small.

The result of Liu and Yang generalizes that of Topping [87, Lemma 1], who assumes that $N = S^2$.

For the Yamabe function, an optimal Łojasiewicz–Simon gradient inequality, has been obtained by Carlotto, Chodosh, and Rubinstein [18] under the hypothesis that the critical point is *integrable* in the sense of their [18, Definition 8], a condition that they observe in [18, Lemma 9] (quoting [1, Lemma 1] due to Adams and Simon) is equivalent to a function on Euclidean space given by the *Lyapunov-Schmidt reduction* of $\mathscr{E}$ being constant on an open neighborhood of the critical point.

For the Yang–Mills energy function for connections on a principal $U(n)$-bundle over a closed Riemann surface, an optimal Łojasiewicz–Simon

Łojasiewicz–Simon inequalities for Morse–Bott functions IV

gradient inequality, has been obtained by Råde [74, Proposition 7.2] when the Yang–Mills connection is *irreducible*.

Given the desirability of treating an energy function as a *Morse function* whenever possible, for example in the spirit of Atiyah and Bott [6] for the Yang–Mills equation over Riemann surfaces, it is useful to rephrase these integrability conditions in the spirit of Morse theory.

Łojasiewicz–Simon inequalities for Morse–Bott functions V

Definition 3.5 (Morse–Bott function)

(See Austin and Braam [7, Section 3.1].) Let $\mathscr{B}$ be a smooth Banach manifold, $\mathscr{E} : \mathscr{B} \to \mathbb{R}$ be a C^2 function, and $\operatorname{Crit} \mathscr{E} := \{x \in \mathscr{B} : \mathscr{E}'(x) = 0\}$. A smooth submanifold $\mathscr{C} \hookrightarrow \mathscr{B}$ is called a *nondegenerate critical submanifold of* $\mathscr{E}$ if $\mathscr{C} \subset \operatorname{Crit} \mathscr{E}$ and

$$(T\mathscr{C})_x = \operatorname{Ker} \mathscr{E}''(x), \quad \forall\, x \in \mathscr{C}, \tag{23}$$

where $\mathscr{E}''(x) : (T\mathscr{B})_x \to (T\mathscr{B})_x^*$ is the Hessian of $\mathscr{E}$ at the point $x \in \mathscr{C}$. One calls $\mathscr{E}$ a *Morse–Bott function* if its critical set $\operatorname{Crit} \mathscr{E}$ consists of nondegenerate critical submanifolds.

Łojasiewicz–Simon inequalities for Morse–Bott functions VI

Definition 3.5 (Morse–Bott function)

We say that a C^2 function $\mathscr{E} : \mathscr{B} \to \mathbb{R}$ is *Morse–Bott at a point* x_0 if there is an open neighborhood $U \subset \mathscr{B}$ of x_0 such that $U \cap \operatorname{Crit} \mathscr{E}$ is a (relatively open) smooth submanifold of $\mathscr{B}$ and (23) holds at x_0.

Definition 3.5 is a restatement of definitions of a Morse–Bott function on a finite-dimensional manifold, but we omit the condition that $\mathscr{C}$ be compact and connected as in Nicolaescu [68, Definition 2.41] or the condition that $\mathscr{C}$ be compact in Bott [14, Definition, p. 248].

Note that if $\mathscr{B}$ is a Riemannian manifold and $\mathscr{N}$ is the normal bundle of $\mathscr{C} \hookrightarrow \mathscr{B}$, so $\mathscr{N}_x = (T\mathscr{C})_x^\perp$ for all $x \in \mathscr{C}$, where $(T\mathscr{C})_x^\perp$ is the orthogonal complement of $(T\mathscr{C})_x$ in $(T\mathscr{B})_x$, then (23) is equivalent to the assertion that the restriction of the Hessian to the fibers of the normal bundle of $\mathscr{C}$,

$$\mathscr{E}''(x) : \mathscr{N}_x \to (T\mathscr{B})_x^*,$$

Łojasiewicz–Simon inequalities for Morse–Bott functions VII

is *injective* for all $x \in \mathscr{C}$; using the Riemannian metric on $\mathscr{B}$ to identify $(T\mathscr{B})^*_x \cong (T\mathscr{B})_x$, we see that $\mathscr{E}''(x) : \mathscr{N}_x \cong \mathscr{N}_x$ is an isomorphism for all $x \in \mathscr{C}$.

In other words, the condition (23) is equivalent to the assertion that the Hessian of $\mathscr{E}$ is an isomorphism of the normal bundle $\mathscr{N}$ when $\mathscr{B}$ has a Riemannian metric.

The Yang–Mills energy function for connections on a principal G-bundle over X is Morse–Bott when X is a closed Riemann surface — see the article by Atiyah and Bott [6] and the discussion by Swoboda [82, p. 161].

However, it appears difficult to extend this result to the case where X is a closed four-dimensional Riemannian manifold.

Łojasiewicz–Simon inequalities for Morse–Bott functions VIII

However, given a Morse–Bott energy function, we then have the

Theorem 3.6 (Optimal Łojasiewicz–Simon gradient inequality for Morse–Bott functions on Banach spaces)

(See F. and Maridakis [30, Theorem 4].) Assume the hypotheses of Theorem 3.3 or of Theorem 3.4. If $\mathscr{M}$ is C^1 and $\mathscr{E}$ is Morse–Bott function at x_∞ in the sense of Definition 3.5, then the conclusions of Theorem 3.3 or 3.4 hold with $\theta = 1/2$.

Theorem 3.6 was first proved by Chill [20, Corollary 3.12], [21, Corollary 4] and Haraux and Jendoubi [41, Theorem 2.1] but under the additional hypothesis that $\mathscr{X}$ is a Hilbert space and $\mathscr{H} = \mathscr{X}$, although in less general settings it is originally due to Simon [80, Sections 3.12 and 3.13].

Morse–Bott property of analytic functions on Banach Spaces with Łojasiewicz exponent 1/2

Łojasiewicz exponent 1/2 implies Morse–Bott property of analytic functions on Banach spaces

Łojasiewicz exponent 1/2 implies Morse–Bott property I

It is a consequence of the Morse–Bott Lemma (see forthcoming Theorems 4.6) that a C^2 Morse–Bott function on an open neighborhood of a critical point in a Banach space obeys a Łojasiewicz gradient inequality with the optimal exponent one half.

We prove converses (see forthcoming Theorems 6 and Corollary 7) for analytic functions on Banach spaces: *If the Łojasiewicz exponent of an analytic function is equal to one half at a critical point, then the function is Morse–Bott and thus its critical set nearby is an analytic submanifold.*

The main ingredients in our proofs are the Łojasiewicz gradient inequality for an analytic function on a finite-dimensional vector space [59] and the Morse Lemma (see forthcoming Theorem 8) for functions on Banach spaces with degenerate critical points that generalize previous versions in the literature, and which we also use to give streamlined proofs of the

Łojasiewicz exponent 1/2 implies Morse–Bott property II

Łojasiewicz–Simon gradient inequalities for analytic functions on Banach spaces (see forthcoming Theorem 9).

If $\mathscr{U} \subset \mathscr{X}$ is an open subset, $f : \mathscr{U} \to \mathbb{K}$ is a C^2 function, and $\operatorname{Crit} f = \{x \in \mathscr{U} : f'(x) = 0\}$ is a C^2, connected submanifold of $\mathscr{U}$, then the tangent space, $T_x \operatorname{Crit} f$, is contained in $\operatorname{Ker} f''(x)$, for each $x \in \operatorname{Crit} f$, where $f'(x) \in \mathscr{X}^*$ and $f''(x) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \mathscr{X}^*)$.

Łojasiewicz exponent 1/2 implies Morse–Bott property III

Definition 4.1 (Morse–Bott properties)

Let $\mathscr{X}$ be a Banach space over $\mathbb{K}$, and $\mathscr{U} \subset \mathscr{X}$ be an open neighborhood of the origin, and $f : \mathscr{U} \to \mathbb{K}$ be a C^2 function such that $\operatorname{Crit} f$ is a C^2, connected submanifold.

1. If $x_0 \in \operatorname{Crit} f$ and $\operatorname{Ker} f''(x_0) \subset \mathscr{X}$ has a closed complement $\mathscr{X}_0$ and $\operatorname{Ran} f''(x_0) = \mathscr{X}_0^*$, and $T_{x_0} \operatorname{Crit} f = \operatorname{Ker} f''(x_0)$, then f is *Morse–Bott at the point* x_0;
2. If f is *Morse–Bott at each point* $x \in \operatorname{Crit} f$, then f is *Morse–Bott along* $\operatorname{Crit} f$ or a *Morse–Bott function*.

The Morse–Bott Lemma (see Theorem 4.6) implies that if f is Morse–Bott at a point, as in Definition 4.1 (1), then f is Morse–Bott along an open neighborhood of that point in $\operatorname{Crit} f$, as in Definition 4.1 (2).

Łojasiewicz exponent 1/2 implies Morse–Bott property IV

When $\mathscr{X}$ is infinite-dimensional, then one must impose hypotheses on f in addition to those of Bott in the finite-dimensional case in order to obtain a tractable version, such as Theorem 4.6, of the classical Morse–Bott Lemma.

We can now state the

Theorem 6 (Morse–Bott property of an analytic function with Łojasiewicz exponent one half)

Let $\mathscr{X}$ be a Banach space over $\mathbb{K}$, and $\mathscr{U} \subset \mathscr{X}$ be an open neighborhood of the origin, and $f : \mathscr{U} \to \mathbb{K}$ be an analytic function such that $f(0) = 0$ and $f'(0) = 0$ and $f''(0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is a Fredholm operator with index zero. If there is a constant $C \in (0, \infty)$ such that, after possibly shrinking $\mathscr{U}$,

$$\|f'(x)\|_{\mathscr{X}^*} \geq C|f(x)|^{1/2}, \quad \forall\, x \in \mathscr{U}, \tag{24}$$

then f is a Morse–Bott function in the sense of Definition 4.1.

Łojasiewicz exponent 1/2 implies Morse–Bott property V

The conclusion of Theorem 6 implies that (using the Morse–Bott Lemma provided by Theorem 4.6), after possibly shrinking $\mathscr{U}$, there are an open neighborhood, $\mathscr{V} \subset \mathscr{X}$, of the origin and an analytic diffeomorphism, $\Phi : \mathscr{V} \to \mathscr{U}$, such that $\Phi(0) = 0$ and

$$f(\Phi(y)) = \frac{1}{2}\langle y, Ay\rangle, \quad \forall\, y \in \mathscr{V},$$

where $A = (f \circ \Phi)''(0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \mathscr{X}^*)$ and, letting $\mathscr{K} := \operatorname{Ker} A \subset \mathscr{X}$ denote the finite-dimensional kernel of A with closed complement, $\mathscr{X}_0 \subset \mathscr{X}$, and $\operatorname{Ran} A = \mathscr{X}_0^* \subset \mathscr{X}^*$ denote the closed range of A with finite-dimensional complement, $\mathscr{K}^*$, in $\mathscr{X}^* = \mathscr{X}_0^* \oplus \mathscr{K}^*$, we have

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} : \mathscr{X}_0 \oplus \mathscr{K} \to \mathscr{X}_0^* \oplus \mathscr{K}^*,$$

where $A_0 \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}_0, \mathscr{X}_0^*)$ is an isomorphism of Banach spaces.

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Łojasiewicz exponent 1/2 implies Morse–Bott property VI

Thus, $(f \circ \Phi)'(y) = Ay$, for all $y \in \mathscr{V}$, and $\operatorname{Crit} f \circ \Phi = \mathscr{V} \cap \operatorname{Ker} A$, an analytic submanifold of $\mathscr{V}$ of dimension equal to $\dim \operatorname{Ker} A$.

When we specialize Theorem 6 to $\mathscr{X} = \mathbb{K}^d$, we obtain the desired characterization of the optimal Łojasiewicz exponent for analytic functions on finite-dimensional vector spaces.

Corollary 7 (Morse–Bott property of an analytic function with Łojasiewicz exponent one half)

Let $d \geq 1$ be an integer, $U \subset \mathbb{K}^d$ be an open neighborhood of the origin, and $f : U \to \mathbb{K}$ be a non-constant analytic function such that $f(0) = 0$ and $f'(0) = 0$. If there is a constant $C \in (0, \infty)$ such that, after possibly shrinking U, the function f obeys the Łojasiewicz gradient inequality (3) with exponent $\theta = 1/2$, then f is a Morse–Bott function.

Łojasiewicz exponent 1/2 implies Morse–Bott property VII

Corollary 7 has a simpler interpretation than Theorem 6.

By the Morse–Bott Lemma (Theorem 4.6 with $\mathscr{X} = \mathbb{K}^d$ and diagonalization [47, p. 278] of the symmetric matrix A over $\mathbb{K}$), after possibly shrinking U, there are an open neighborhood, $V \subset \mathbb{K}^d$, of the origin and an analytic diffeomorphism, $\Phi : V \to U$, such that $\Phi(0) = 0$ and

$$f(\Phi(y)) = \frac{1}{2}\sum_{i=1}^{d-c} a_i y_i^2, \quad \forall\, y \in V,$$

where $a_i \in \mathbb{K} \setminus \{0\}$, for $i = 1, \ldots, d-c$ and an integer c obeying $0 \leq c \leq d-1$.

Thus, $(f \circ \Phi)'(y) = (a_1 y_1, \ldots, a_{d-c} y_{d-c}, 0, \ldots, 0) \in \mathbb{K}^d$, for all $y \in V$, and $\operatorname{Crit} f \circ \Phi = \{y \in V : y_1 = \cdots = y_{d-c} = 0\}$, an analytic submanifold of V of dimension c.

Morse Lemma for functions on Banach spaces with degenerate critical points

Morse Lemma for degenerate critical points I

It is important to carefully distinguish between the *Morse–Bott Lemma* and the more general *Morse Lemma for functions on Banach spaces with degenerate critical points* (also known as the *Morse Lemma with parameters* or *Splitting Lemma*).

The latter makes no assumption on whether the critical set is a submanifold or, even if it is a submanifold, whether its tangent space at each critical point is equal to the kernel of the Hessian operator at that point.

Morse Lemma for degenerate critical points II

Theorem 8 (Morse Lemma for functions on Banach spaces with degenerate critical points)

Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces over $\mathbb{K}$, and $\mathscr{U} \subset \mathscr{X}$ and $\mathscr{V} \subset \mathscr{Y}$ be open neighborhoods of the origin, and $f : \mathscr{X} \times \mathscr{Y} \supset \mathscr{U} \times \mathscr{V} \ni (x, y) \mapsto f(x, y) \in \mathbb{K}$ be a C^{p+2} function ($p \geq 1$) such that $f(0, 0) = 0$ and $D_x f(0, 0) = 0$. If $D_x^2 f(0, 0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is invertible then, after possibly shrinking $\mathscr{U}$ and $\mathscr{V}$, there are an open neighborhood $\mathscr{U}'$ of the origin and a C^p diffeomorphism, $\Phi : \mathscr{U}' \times \mathscr{V} \ni (z, y) \mapsto (x, y) \in \mathscr{U} \times \mathscr{V}$ with $\Phi(0, 0) = (0, 0)$ and

$$D\Phi(0,0) = \begin{pmatrix} \mathrm{id}_{\mathscr{X}} & \star \\ 0 & \mathrm{id}_{\mathscr{Y}} \end{pmatrix} \in \mathscr{L}(\mathscr{X} \oplus \mathscr{Y}) \tag{25}$$

Morse Lemma for degenerate critical points III

Theorem 8 (Morse Lemma for functions on Banach spaces with degenerate critical points)

such that

$$f(\Phi(z,y)) = f(\Phi(0,y)) + \frac{1}{2}\langle z, Az\rangle, \quad \forall\,(z,y) \in \mathscr{U}' \times \mathscr{V}, \tag{26}$$

where $A := D_x^2 f(0,0) = D_z^2(f\circ\Phi)(0,0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \mathscr{X}^*)$. *If* f *is analytic, then* Φ *is analytic.*

Morse Lemma for degenerate critical points IV

Remark 4.2 (Previous versions of the Morse Lemma for functions with degenerate critical points)

Theorem 8 is a generalization of Hörmander [46, Lemma C.6.1] from the case where f is C^∞ and $\mathscr{H} = \mathbb{R}^d$ with its standard inner product; and [38, Lemma 1], due to Gromoll and Meyer, from the case where f is C^∞ and $\mathscr{H}$ is a real separable Hilbert space and the Hessian is Fredholm at the critical point; and [63, Theorem 1], due to Mawhin and Willem, from the case where f is C^2 and $\mathscr{H}$ is a real Hilbert space and the Hessian is Fredholm at the critical point.

Morse Lemma for degenerate critical points V

Remark 4.3 (Previous versions of the Morse Lemma for functions on Banach spaces)

Palais proved the Morse Lemma for smooth functions on Hilbert spaces in [70, p. 307] and later extended and simplified his proof to give the Morse Lemma for smooth functions on Banach spaces in [71, p. 968]). Gromoll and Meyer [38, Lemma 1] consider C^∞ functions on Hilbert spaces with Fredholm Hessian operator while Mawhin and Willem [63, Theorem 1] relax the regularity requirement of Gromoll and Meyer in [38] from C^∞ to C^2.

Morse Lemma for degenerate critical points VI

Remark 4.4 (Regularity of the function f)

It is likely that one adapt the arguments of Cambini [17], Kuiper [50], and Mawhin and Willem [63] to reduce the C^{p+2} (with $p \geq 1$) regularity requirement on f in Theorem 8 to C^2, as those authors allow for their versions of the Morse Lemma on Banach spaces.

Morse Lemma for degenerate critical points VII

Remark 4.5 (Morse Lemma for functions with degenerate critical points and Lyapunov-Schmidt reduction)

Theorem 8 may be regarded as a more refined version of the technique of Lyapunov-Schmidt reduction (see Guo and Wu [39, Section 5.1], Huang [48, Proposition 2.4.1], or Nirenberg [69, Section 2.7.6]) when $\mathscr{Y}$ has finite dimension.
Theorem 8 reduces the proof of the Łojasiewicz gradient inequality for analytic functions on Banach spaces to the Łojasiewicz gradient inequality for analytic functions on Euclidean space.

Morse–Bott Lemma for functions on Banach spaces I

Our Morse Lemmas for functions on Banach spaces with degenerate critical points (Theorem 8) lead to useful Morse–Bott Lemmas.

Theorem 4.6 (Morse–Bott Lemma for functions on Banach spaces)

Let $\mathscr{X}$ be a Banach space over $\mathbb{K}$, and $\mathscr{U} \subset \mathscr{X}$ be an open neighborhood of the origin, and $f : \mathscr{U} \to \mathbb{K}$ be a C^{p+2} function ($p \geq 1$) such that $f(0) = 0$. If f is Morse–Bott at the origin in the sense of Definition 4.1 (1), then, after possibly shrinking $\mathscr{U}$, there are an open neighborhood of the origin, $\mathscr{V} \subset \mathscr{X}$, and a C^p diffeomorphism, $\mathscr{V} \ni y \mapsto x = \Phi(y) \in \mathscr{U}$ with $\Phi(0) = 0$ and $D\Phi(0) = \mathrm{id}_{\mathscr{X}}$, such that

$$f(\Phi(y)) = \frac{1}{2}\langle y, Ay\rangle, \quad \forall\, y \in \mathscr{V}, \tag{27}$$

where $A := f''(0) = (f \circ \Phi)''(0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \mathscr{X}^*)$. If f is analytic, then Φ is analytic.

Gradient inequalities for functions on Banach spaces I

Theorem 8 can be used to give an alternative proof of the Łojasiewicz–Simon gradient inequalities for analytic functions on Banach spaces which complement those of F and Maridakis in [30].

Theorem 9 (Łojasiewicz–Simon gradient inequality for analytic functions on Banach spaces)

(Compare F and Maridakis [30, Theorem 1].) Let $\mathscr{X}$ be a Banach space over $\mathbb{K}$, and $\mathscr{U} \subset \mathscr{X}$ be an open neighborhood of the origin, and $f : \mathscr{U} \to \mathbb{K}$ be an analytic function such that $f(0) = 0$ and $f'(0) = 0$. If $f''(0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is a Fredholm operator with index zero then, after possibly shrinking $\mathscr{U}$, there are constants $C \in (0, \infty)$ and $\theta \in [1/2, 1)$ such that

$$\|f'(x)\|_{\mathscr{X}^*} \geq C|f(x)|^{\theta}, \quad \forall\, x \in \mathscr{U}. \tag{28}$$

Informal survey of applications of the Łojasiewicz gradient inequality in gauge theory and geometric analysis

Råde's work on retractions of quotient spaces of connections onto subspaces of Yang–Mills connections over closed manifolds of dimension $d = 2$ or 3

Retractions of spaces of connections I

Let G be a compact semisimple Lie group and P a smooth principal G-bundle over a closed smooth Riemannian manifold (X, g) of dimension $d = 2$ or 3. The Yang–Mills energy function is

$$\mathscr{YM}(A) := \int_X |F_A|^2 \, d\,\mathrm{vol}_g, \tag{29}$$

- where $F_A := d_A \circ d_A \in \Omega^2(\mathrm{ad}P)$ is the curvature of a connection $A \in \mathscr{A}(P) = A_{\mathrm{REF}} + \Omega^1(X; \mathrm{ad}P)$,
- $\mathrm{ad}P := P \times_{\mathrm{Ad}} \mathfrak{g}$ is the real vector bundle associated to P by the adjoint representation of G on its Lie algebra,
- $d_A : \Omega^i(\mathrm{ad}P) \to \Omega^{i+1}(\mathrm{ad}P)$ is exterior covariant derivative, and the
- pointwise norm $|\cdot|$ is defined by g and the fiber metric on $\mathrm{ad}P$ defined by the Killing form on $\mathfrak{g}$.

Retractions of spaces of connections II

This energy function is invariant under the action of gauge transformations $u \in \mathrm{Aut}(P)$ on $\mathscr{A}(P)$, so

$$\mathscr{YM} : \mathscr{A}(P) \to [0, \infty)$$

descends to a function

$$\mathscr{YM} : \mathscr{B}(P) \to [0, \infty)$$

on the quotient $\mathscr{B}(P) := \mathscr{A}(P)/\mathrm{Aut}(P)$.

The gradient of $\mathscr{YM}$ with respect to the L^2 metric on $\mathscr{A}(P)$ is given by

$$\mathrm{grad}\,\mathscr{YM}(A) \equiv \mathscr{M}_g(A) = d_A^* F_A \in \Omega^1(\mathrm{ad}P),$$

and critical points of $\mathscr{YM}$ are called *Yang–Mills connections*.

Retractions of spaces of connections III

For any constant $E > 0$, fundamental results of Karen Uhlenbeck [88] imply that the subspace of $\mathscr{B}(P)$ represented by Yang–Mills connections A with energy $\mathscr{YM}[A] \leq E$ is *compact* when $d = 2$ or 3.

With respect to suitable choices of Sobolev space completions of $\Omega^1(\mathrm{ad}P)$, Råde [73, 74] showed that $\mathscr{YM}$ is real analytic and obeys a Łojasiewicz–Simon gradient inequality,

$$\|\operatorname{grad}\mathscr{YM}(A)\|_{W^{-1,2}(X)} \geq |\mathscr{YM}(A) - \mathscr{YM}(A_\infty)|^\theta,$$

for $\theta \in [\frac{1}{2}, 1)$ and all A close to a Yang–Mills connection A_∞.

Råde noted that this implies (for $d = 2$ or 3) that the energies of Yang–Mills connections form a *discrete subset* of $\mathbb{R}$ with no accumulation points (except at ∞), so $0 \leq \lambda_0 < \lambda_1 < \cdots < \infty$.

Retractions of spaces of connections IV

Using his Łojasiewicz–Simon gradient inequality Råde proved that, for any $W^{1,2}$ initial connection A_0 on P, the gradient flow $A(t)$ for $\mathscr{YM}$ exists for all $t \in [0, \infty)$ and converges to a Yang–Mills connection on P as $t \to \infty$.

Consequently, he proved that Yang–Mills gradient flow defines retractions of $\mathscr{A}(P)$ and $\mathscr{B}(P)$ onto unions of subspaces of (gauge-equivalence classes of) Yang–Mills connections.

Łojasiewicz–Simon gradient inequality for the Yang–Mills energy function

Łojasiewicz–Simon for Yang–Mills connections I

There has long been interest in generalizing results of Råde for dimension $d = 2$ or 3, along with those of Donaldson [22], Kozono, Maeda, and Naito [49], Struwe [81] for $d = 4$ in special cases (who do not use Łojasiewicz–Simon gradient inequalities), to dimension $d \geq 4$.

As an application of Theorem 3.1, we have the following generalization of Råde's [74, Proposition 7.2] from X of dimension $d = 2$ or 3.

Theorem 5.1 (Łojasiewicz–Simon inequality for the Yang–Mills function)

(See F., [25, Theorems 23.1 & 23.17]; (F. and Maridakis, [32, Theorem 4].) Let (X, g) be a closed, Riemannian, smooth manifold of dimension d, and G be a compact Lie group, and P be a smooth principal G-bundle over X. Let A_1 be a connection of class C^∞ on P, and A_∞ a Yang–Mills connection for g of class $W^{1,q}$, with $q \in [2, \infty)$ obeying $q > d/2$. If $p \in [2, \infty)$ obeys $d/2 \leq p \leq q$

Łojasiewicz–Simon for Yang–Mills connections II

Theorem 5.1 (Łojasiewicz–Simon inequality for the Yang–Mills function)

then the gradient map,

$$\mathscr{M}_g : W^{1,p}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P) \to W^{-1,p}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P),$$

is real analytic and there are positive constants $Z \in [1, \infty)$, $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$, depending on A_1, A_∞,, g, G, p, and q with the following significance. If A is a $W^{1,q}$ connection on P obeying

$$\|A - A_\infty\|_{W^{1,p}_{A_1}(X)} < \sigma, \tag{30}$$

then the Yang–Mills energy function (29) obeys

$$\|\mathscr{M}_g(A)\|_{W^{-1,p}_{A_1}(X)} \geq Z|\mathscr{E}_g(A) - \mathscr{E}_g(A_\infty)|^\theta. \tag{31}$$

The proof of Theorem 5.1 has two main ingredients:

Łojasiewicz–Simon for Yang–Mills connections III

Prove a slice theorem that says if

$$\|A - A_\infty\|_{W^{1,p}_{A_1}(X)},$$

then there is a $W^{2,q}$ gauge transformation $u \in \mathrm{Aut}(P)$ such that

$$d^*_{A_\infty}(u(A) - A_\infty) = 0$$

and

$$\|u(A) - A_\infty\|_{W^{1,p}_{A_1}(X)} \leq C\|A - A_\infty\|_{W^{1,p}_{A_1}(X)}.$$

This standard when $p > d/2$, but difficult when $p = d/2$ (the case we want when $d = 4$) because $W^{2,\frac{d}{2}}(X) \not\subset C(X)$ and is not a Banach algebra.

Łojasiewicz–Simon for Yang–Mills connections IV

Apply the abstract Łojasiewicz–Simon gradient inequality (Theorem 3.1) to the Yang–Mills energy function, noting that the restriction of the Hessian $\mathscr{M}'(A_\infty)$ to the slice $\operatorname{Ker} d^*_{A_\infty} \cap W^{1,p}_{A_1}(X; \Lambda^1 \otimes \operatorname{ad} P)$ takes the form

$$\mathscr{M}'(A_\infty) = d^*_{A_\infty} d_{A_\infty} + d_{A_\infty} d^*_{A_\infty} + \text{lower-order terms},$$

and can be shown to be Fredholm with index zero, as required.

Application to energy gaps for Yang–Mills connections

Using our Theorem 5.1, one can extend classical energy gap results due to Bourguignon, Lawson, and Simons [15, 16], Gerhardt [36], Min-Oo [64], Nakajima [65], Parker [72], and others.

Łojasiewicz–Simon for Yang–Mills connections V

Theorem 5.2 ($L^{d/2}$-energy gap for Yang–Mills connections)

Let G be a compact Lie group and P be a principal G-bundle over a closed, smooth manifold, X, of dimension $d \geq 2$ and endowed with a smooth Riemannian metric, g. Then there is a positive constant, $\varepsilon = \varepsilon(d, g, G) \in (0, 1]$, with the following significance. If A is a smooth Yang–Mills connection on P whose curvature, F_A, obeys

$$\|F_A\|_{L^{d/2}(X)} \leq \varepsilon, \tag{32}$$

then A is a flat connection.

Łojasiewicz–Simon for Yang–Mills connections VI

Theorem 5.3 (Relative $W^{1,2}$ energy gap for Yang–Mills connections over low-dimensional manifolds)

Let G be a compact Lie group and P be a principal G-bundle over a closed, smooth, Riemannian manifold, (X, g), of dimension $d = 2, 3$, or 4. If A_∞ is Yang–Mills connection on P, then there is a positive constant, $\varepsilon = \varepsilon(A_\infty, d, g, G) \in (0, 1]$, with the following significance. If A is another smooth Yang–Mills connection on P, and

$$\|u(A) - A_\infty\|_{W^{1,2}_{A_\infty}(X)} < \varepsilon, \tag{33}$$

for some $u \in \mathrm{Aut}(P)$, then $\mathscr{E}(A) = \mathscr{E}(A_\infty)$.

Global existence and convergence of Yang–Mills gradient flow near a local minimum

Yang–Mills gradient flow over four-dimensional manifolds I

The forthcoming conjecture essentially goes back to Atiyah and Bott [6], Sedlacek [77], Taubes [83, 84, 85], and Uhlenbeck [88, 89]; it appears explicitly in an article by Schlatter, Struwe, and Tahvildar-Zadeh [75, p. 118] and elsewhere.

Over manifolds X of dimension $d = 2$ or 3, the natural L^2 energy bound on the curvature F_A of a connection A on a principal G-bundle over X yields compactness and control of the local connection one-forms by fundamental results of Karen Uhlenbeck [88].

When $d = 4$, limits of sequences of connections with L^2-bounded curvature can exhibit point singularities and *energy bubbling*: one has a weaker notion of compactness due to Uhlenbeck [88, 89], Sedlacek [77], and Donaldson [24].

Yang–Mills gradient flow over four-dimensional manifolds II

When $d \geq 5$, limits of sequences of connections with L^2-bounded curvature can exhibit far worse singuarities (see Tian [86]).

Conjecture 5.4 (Global existence and convergence of Yang–Mills gradient flow over closed four-dimensional manifolds)

Let G be a compact Lie group and P a principal G-bundle over a closed, connected, four-dimensional, smooth manifold, X, with Riemannian metric, g. If A_0 is a smooth connection on P, then there is a smooth solution, $A(t)$ for $t \in [0, \infty)$, to the gradient flow,

$$\frac{\partial A}{\partial t} = -d^{*_g}_{A(t)} F_{A(t)}, \tag{34a}$$

$$A(0) = A_0, \tag{34b}$$

for the Yang–Mills energy function (29) with respect to the L^2 Riemannian metric on the affine space of connections on P. Moreover, as $t \to \infty$, the flow, $A(t)$, converges to a smooth Yang–Mills connection, A_∞, on P.

Global existence and convergence of Yang–Mills flow I

Theorem 5.5 (Global existence and convergence of Yang–Mills gradient flow near a local minimum)

(See F., [25, Theorem 6].) Let G be a compact Lie group and P a principal G-bundle over a closed, connected, oriented, smooth manifold, X, of dimension $d \geq 2$ and with Riemannian metric, g. Let A_1 and $A_{\min}$ be C^∞ connections on P, with $A_{\min}$ being a local minimum, and let (a) $k = 1$ and $p = 2$ if $2 \leq d \leq 4$, or (b) $k = 2$ and $p > d/2$ if $d \geq 5$. Then there are constants $c \in [1, \infty)$, and $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$, depending on $(A_1, A_{\min}, g, p)$, with the following significance.

Global existence and convergence of Yang–Mills flow II

Theorem 5.5 (Global existence and convergence of Yang–Mills gradient flow near a local minimum)

1. *Global existence:* *There is a constant* $\varepsilon \in (0, \sigma/4)$, *depending on* $(A_1, A_{\min}, g, p)$, *with the following significance. If* A_0 *is a* C^∞ *connection on* P *such that*

$$\|A_0 - A_{\min}\|_{W^{k,p}_{A_1}(X)} < \varepsilon,$$

then there exists a solution, $A(t) = A_0 + a(t)$ *for* $t \in [0, \infty)$, *with*

$$a \in C^\infty([0, \infty) \times X; \Lambda^1 \otimes \mathrm{ad}P),$$

to the Yang–Mills gradient flow (34) *with initial data,* $A(0) = A_0$, *and*

$$\|A(t) - A_{\min}\|_{W^{k,p}_{A_1}(X)} < \sigma/2, \quad \forall\, t \in [0, \infty).$$

Global existence and convergence of Yang–Mills flow III

Theorem 5.5 (Global existence and convergence of Yang–Mills gradient flow near a local minimum)

2. *Dependence on initial data:* *The solution, $A(t)$ for $t \in [0,\infty)$, varies continuously with respect to A_0 in the $C_{\mathrm{loc}}([0,\infty); W^{k,p}_{A_1}(X;\Lambda^1 \otimes \mathrm{ad}P))$ topology and, more generally, smoothly for all non-negative integers, k, l, in the $C^l_{\mathrm{loc}}([0,\infty); H^m_{A_1}(X;\Lambda^1 \otimes \mathrm{ad}P))$ topology.*
3. *Convergence:* *As $t \to \infty$, the flow, $A(t)$, converges strongly with respect to the norm on $W^{k,p}_{A_1}(X;\Lambda^1 \otimes \mathrm{ad}P)$ to a Yang–Mills connection, A_∞, of class C^∞ on P, and the gradient-flow line has finite length in the sense that*

$$\int_0^\infty \left\| \frac{\partial A}{\partial t} \right\|_{W^{k,p}_{A_1}(X)} dt < \infty.$$

If $A_{\min}$ is a cluster point of the orbit, $O(A) = \{A(t) : t \geq 0\}$, then $A_\infty = A_{\min}$.

Global existence and convergence of Yang–Mills flow IV

Theorem 5.5 (Global existence and convergence of Yang–Mills gradient flow near a local minimum)

4. *Convergence rate:* *For all* $t \geq 1$,

$$\|A(t) - A_\infty\|_{W^{k,p}_{A_1}(X)} \leq \begin{cases} \dfrac{1}{c(1-\theta)} \left(c^2(2\theta - 1)(t-1) + (\mathscr{E}(A_0) - \mathscr{E}(A_\infty))^{1-2\theta} \right)^{-(1-\theta)/(2\theta-1)}, \\ \dfrac{2}{c} \sqrt{\mathscr{E}(A_0) - \mathscr{E}(A_\infty)} \exp(-c^2(t-1)/2), \end{cases}$$

for $1/2 < \theta < 1$ *and* $\theta = 1/2$, *respectively.*

5. *Stability:* *As an equilibrium of the Yang–Mills gradient flow* (34), *the point* A_∞ *is Lyapunov stable; if* A_∞ *is isolated or a cluster point of the orbit* $O(A)$, *then* A_∞ *is uniformly asymptotically stable.*

Global existence and convergence of Yang–Mills flow V

Theorem 5.5 (Global existence and convergence of Yang–Mills gradient flow near a local minimum)

6. *Uniqueness:* *Any two solutions are equivalent modulo a path of gauge transformations,*

$$u \in C^\infty([0,\infty) \times X; \mathrm{Ad}P), \quad u(0) = \mathrm{id}_P.$$

Some applications, corollaries, and examples

Let $b^{\pm}(X)$ denote the dimension of the vector spaces of self-dual and anti-self-dual harmonic two-forms on X, respectively.

Global existence and convergence of Yang–Mills flow VI

Theorem 5.6 (Long-time existence and convergence of Yang–Mills gradient flow started with almost minimal energy when $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$)

Let G be a compact Lie group and P a principal G-bundle over a closed, connected, Riemannian, smooth four-dimensional manifold, X, and $K \in [1, \infty)$ and $p > 2$ constants. Then there is a positive constant, ε, with the following significance. Assume that

1. *The Riemannian metric on X is generic and $\pi_1(X)$ is trivial;*
2. *A_0 has almost minimal energy in the sense that $\|F_{A_0}^+\|_{L^p(X)} < \varepsilon$;*

and $\|F_{A_0}\|_{L^p(X)} \leq K$ and one of the following holds:

3. *$b^+(X) = 0$ and $G = \mathrm{SU}(2)$ or $G = \mathrm{SO}(3)$; or*
4. *$b^+(X) > 0$, $G = \mathrm{SO}(3)$, and the second Stiefel-Whitney class, $w_2(P) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$, is non-trivial;*

Then there is a smooth solution, $A(t)$ for $t \in [0, \infty)$, to the Yang–Mills gradient flow with $A(0) = A_0$. Moreover, as $t \to \infty$, the flow, $A(t)$, converges to a smooth anti-self-dual connection, A_∞, on P.

Global existence and convergence of Yang–Mills flow VII

For a Riemannian metric g on a four-dimensional, oriented manifold, X, let $R_g(x)$ denote its scalar curvature at $x \in X$ and $\mathscr{W}_g^{\pm}(x) \in \mathrm{End}(\Lambda_x^{\pm})$ its self-dual and anti-self-dual Weyl curvature tensors at x, where $\Lambda_x^2 = \Lambda_x^+ \oplus \Lambda_x^-$ and $\Lambda_x^2 = \Lambda^2(T_x^*X)$. Set

$$w_g^{\pm}(x) := \text{Largest eigenvalue of } \mathscr{W}_g^{\pm}(x), \quad \forall\, x \in X.$$

We call g *positive* if there is a metric $\tilde{g}$, conformally equivalent to g (so $\tilde{g} = e^{2f} g$ for some $f \in C^\infty(X)$), such that

$$\frac{1}{3} R_{\tilde{g}} - 2 w_{\tilde{g}}^+ > 0 \quad \text{on } X,$$

that is, $R_{\tilde{g}}/3 - 2w_{\tilde{g}}^+ \in \mathrm{End}(\Lambda^+)$ is positive definite. The simplest example is $X = S^4$ with its standard round metric.

Global existence and convergence of Yang–Mills flow VIII

Theorem 5.7 (Long-time existence and convergence of Yang–Mills gradient flow started with almost minimal energy and positive metric)

Let G be a compact Lie group and P a principal G-bundle over a closed, connected, Riemannian, smooth four-dimensional manifold, X, and $K \in [1, \infty)$ and $p > 2$ constants. Then there is a positive constant, ε, with the following significance. Assume that:

1. *The Riemannian metric on X is positive;*
2. *A_0 has almost minimal energy in the sense that $\|F^+_{A_0}\|_{L^p(X)} < \varepsilon$;*

and $\|F_{A_0}\|_{L^p(X)} \leq K$. Then there is a smooth solution, $A(t)$ for $t \in [0, \infty)$, to the Yang–Mills gradient flow with $A(0) = A_0$. Moreover, as $t \to \infty$, the flow, $A(t)$, converges to a smooth anti-self-dual connection, A_∞, on P.

Global existence and convergence of Yang–Mills flow IX

Remark 5.8

When X has dimension 4, the hypothesis that A_0 be $W^{1,2}$-norm close to a local minimum (for example, an anti-self-dual Yang–Mills connection) can be relaxed to the energy condition

$$\|F_A^{+,g}\|_{L^2(X,g)} < \varepsilon,$$

in the presence of various combinations (as described elsewhere in this talk) of additional hypotheses on G, the topology of P, or the Riemannian metric, g, that guarantee *existence of anti-self-dual Yang–Mills connections* on P.

Global existence and convergence of Yang–Mills flow X

Corollary 5.9 (Existence of flat structures)

Let G be a compact Lie group and P a principal G-bundle over a closed, connected, Riemannian, smooth four-dimensional manifold, X. Then there is a positive constant, ε, with the following significance. If A_0 is a C^∞ connection on P such that

$$\|F_{A_0}\|_{L^2(X)} < \varepsilon,$$

then there is a smooth solution, $A(t)$ for $t \in [0, \infty)$, to the Yang–Mills gradient flow with $A(0) = A_0$. Moreover, as $t \to \infty$, the flow, $A(t)$, converges to a flat connection, Γ_∞, on P.

Global existence and convergence of Yang–Mills flow XI

Corollary 5.10 (Failure of global existence and convergence of Yang–Mills gradient flow)

Let G be a compact Lie group and P a principal G-bundle over a closed, connected, Riemannian, smooth four-dimensional manifold, X. Then there is a positive constant, ε, with the following significance. Suppose there does not exist a Yang–Mills connection, A_{YM}, such that $\|F^+_{A_{\mathrm{YM}}}\|_{L^2(X)} < \varepsilon$. If A_0 is a C^∞ connection on P such that

$$\|F^+_{A_0}\|_{L^2(X)} < \varepsilon,$$

then the solution, $A(t)$ for $t \in [0, T)$, to the Yang–Mills gradient flow with $A(0) = A_0$ must acquire a singularity at $T \in (0, \infty]$.

Morgan–Mrowka–Ruberman's analysis of convergence of anti-self-dual connections with finite L^2 energy over cylinders

Convergence of anti-self-dual connections I

Let P be a principal G-bundle over a closed Riemannian 3-manifold (Y, h).

The cylinder $X = \mathbb{R} \times Y$ is a smooth Riemannian 4-manifold with product metric $g = dt^2 \oplus h$.

Hodge star $\star_g : \Omega^2(X) \to \Omega^2(X)$ defines a linear map such that $\star_g^2 = 1$ and we may write $\Omega^2(X) = \Omega^+(X) \oplus \Omega^-(X)$, where $\Omega^\pm(X)$ are the eigenspaces of $\star_g$ with eigenvalues ± 1.

A connection A on $P \to X$ is *anti-self-dual* or *self-dual* if $F_A^+ = 0$ or $F_A^- = 0$, respectively.

The Bianchi identity ($d_A F_A = 0$) and expression $d_A^* = - \star_g d_A \star_g$ imply that any such connection is Yang–Mills.

Convergence of anti-self-dual connections II

For $X = Y \times \mathbb{R}$, the equation $F_A^+ = 0$ (after applying a suitable gauge transformation) is equivalent to

$$\frac{dB}{dt} = - \star_h F_B,$$

where $B(t)$, for $t \in \mathbb{R}$, is a family of connections on $P \to Y$. One has

$$\operatorname{grad} \mathrm{CS}(B) = \star_h F_B,$$

where the *Chern–Simons function* is given by

$$\mathrm{CS}(B) = \frac{1}{4\pi} \int_Y \operatorname{tr} \left(B \wedge dB + \frac{2}{3} B \wedge B \wedge B \right).$$

Hence, the anti-self-dual equation takes the equivalent form

$$\frac{dB}{dt} = - \operatorname{grad} \mathrm{CS}(B),$$

Convergence of anti-self-dual connections III

and along a solution one has

$$\frac{d}{dt}\,\mathrm{CS}(B) = -\|\operatorname{grad}\mathrm{CS}(B(t))\|^2_{L^2(Y)}.$$

The analysis of such solutions leads to the definition of *Floer homology theory* [23, 35], which plays an important role in the topology of 3-manifolds, knot theory, and the theory of *Donaldson invariants* of smooth 4-manifolds.

Morgan, Mrowka, and Ruberman adapt the method of Leon Simon [78] to prove a Łojasiewicz–Simon gradient inequality for CS on suitably chosen Sobolev completions and prove convergence of $B(t)$ as $t \to \infty$, modulo gauge transformations, to a flat connection Γ on $P \to Y$.

Flat connections on $P \to Y$ are critical points of CS and they serve as generators of the Floer homology groups of Y.

Łojasiewicz–Simon inequalities for harmonic maps I

We describe a consequence of the abstract Łojasiewicz–Simon gradient inequality, Theorem 3.1, for the harmonic map energy function.

Definition 5.11 (Harmonic map energy function)

Let (M, g) and (N, h) be a pair of closed, Riemannian, smooth manifolds. One defines the *harmonic map energy function* by

$$\mathscr{E}_{g,h}(f) := \frac{1}{2}\int_M |df|^2_{g,h}\, d\,\mathrm{vol}_g, \tag{35}$$

for $W^{1,2}$ maps, $f : M \to N$, where $df : TM \to TN$ is the differential.

To define the gradient of the energy function $\mathscr{E}_{g,h}$ in (35) with respect to the L^2 metric on $C^\infty(M; N)$, we first choose an isometric embedding,

Łojasiewicz–Simon inequalities for harmonic maps II

$(N,h) \hookrightarrow \mathbb{R}^n$ for a sufficiently large n (by the Nash isometric embedding theorem [66]), and recall that

$$(u, \mathscr{M}_{g,h}(f))_{L^2(M,g)} = (\mathscr{E}'_{g,h}(f), u)_{L^2(M)} = (\Delta_g f - A_h(df,df), u)_{L^2(M)},$$

for all $u \in C^\infty(M;N)$, that is,

$$\mathscr{M}(f) \equiv \mathscr{M}_{g,h}(f) = \Delta_g f - A_h(df,df).$$

Here, A_h denotes the second fundamental form of the isometric embedding, $(N,h) \hookrightarrow \mathbb{R}^n$ and

$$\Delta_g := -\operatorname{div}_g \operatorname{grad}_g f = -\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^\beta} \left(\sqrt{\det g}\, \frac{\partial f}{\partial x^\alpha} \right)$$

denotes the Laplace-Beltrami operator for (M,g) (with the opposite sign convention to that of [19, Equations (1.14) and (1.33)]) acting on the

Łojasiewicz–Simon inequalities for harmonic maps III

scalar components f^i of $f = (f^1, \ldots, f^n)$ and $\{x^\alpha\}$ denote local coordinates on M.

One says that a smooth map $f : M \to N$ is *harmonic* if it is a critical point of the energy function (35), that is

$$\mathscr{E}'_{g,h}(f) = \Delta_g f - A_h(df, df) = 0.$$

Given a smooth map $f : M \to N$, an isometric embedding $(N, h) \hookrightarrow \mathbb{R}^n$, a non-negative integer k, and $p \in [1, \infty)$, we define the Sobolev norms,

$$\|f\|_{W^{k,p}(M)} := \left(\sum_{i=1}^{n} \|f^i\|^p_{W^{k,p}(M)} \right)^{1/p},$$

with

$$\|f^i\|_{W^{k,p}(M)} := \left(\sum_{j=0}^{k} \int_M |\nabla_g^j f^i|^p \, d\,\mathrm{vol}_g \right)^{1/p},$$

Łojasiewicz–Simon inequalities for harmonic maps IV

where ∇_g denotes the Levi-Civita connection on TM and all associated bundles (such as T^*M and their tensor products), and if $p = \infty$, we define

$$\|f\|_{W^{k,\infty}(M)} = \|f\|_{C^k(M)} := \sum_{i=1}^{n} \sum_{j=0}^{k} \operatorname*{ess\,sup}_M |\nabla_g^j f^i|.$$

If $k = 0$, then we denote $\|f\|_{W^{0,p}(M)} = \|f\|_{L^p(M)}$.

For $p \in [1, \infty)$ and nonnegative integers k, we use [2, Theorem 3.12] (applied to $W^{k,p}(M; \mathbb{R}^n)$ and noting that M is a closed manifold) and Banach space duality to obtain

$$W^{-k,p'}(M; \mathbb{R}^n) = \left(W^{k,p}(M; \mathbb{R}^n)\right)^*,$$

where $p' \in (1, \infty]$ is the dual exponent defined by $1/p + 1/p' = 1$.

Łojasiewicz–Simon inequalities for harmonic maps V

If (N, h) is real analytic, then the isometric embedding $(N, h) \hookrightarrow \mathbb{R}^n$ may also be chosen to be analytic by the analytic isometric embedding theorem due to Nash [67], with a simplified proof due to Greene and Jacobowitz [37]).

As an application of our abstract Łojasiewicz–Simon gradient inequality, Theorem 3.3, we obtain the

Theorem 5.12 (Analyticity and Łojasiewicz–Simon gradient inequality for the harmonic map energy function)

(See [31, Theorem 5].) Let $d \geq 2$ and $k \geq 1$ be integers and $p \in (1, \infty)$ be such that $kp > d$. Let (M, g) and (N, h) be closed, smooth Riemannian manifolds, with M of dimension d. If (N, h) is real analytic (respectively, C^∞) and $f \in W^{k,p}(M; N)$, then the gradient map $\mathscr{M}$ for the energy function, $\mathscr{E} : W^{k,p}(M; N) \to \mathbb{R}$, in (35),

$$W^{k,p}(M; N) \ni f \mapsto \mathscr{M}(f) \in W^{k-2,p}(M; f^* TN) \subset W^{k-2,p}(M; \mathbb{R}^n),$$

is a real analytic (respectively, C^∞) map of Banach spaces.

Łojasiewicz–Simon inequalities for harmonic maps VI

Theorem 5.12 (Analyticity and Łojasiewicz–Simon gradient inequality for the harmonic map energy function)

If (N, h) is real analytic and $f_\infty \in W^{k,p}(M; N)$ is a weakly harmonic map, then there are positive constants $Z \in (0, \infty)$, and $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$, depending on f_∞, g, h, k, p, with the following significance. If $f \in W^{k,p}(M; N)$ obeys

$$\|f - f_\infty\|_{W^{k,p}(M;\mathbb{R}^n)} < \sigma, \tag{36}$$

then the gradient $\mathscr{M}$ of the harmonic map energy function $\mathscr{E}$ in (35) obeys

$$\|\mathscr{M}(f)\|_{W^{k-2,p}(M;f^*TN)} \geq Z|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^\theta. \tag{37}$$

When $W^{k-2,p}(M; \mathbb{R}) \subset L^2(M; \mathbb{R})$, one can improve (37) to

$$\|\mathscr{M}(f)\|_{L^2(M;f^*TN)} \geq Z|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^\theta.$$

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Łojasiewicz–Simon inequalities for harmonic maps VII

Theorem 5.12 extends a version of the Łojasiewicz–Simon gradient inequality that is stated by Simon as [79, Equation (4.27)] and can be derived from his more general [78, Theorem 3], with (36) replaced by

$$\|f - f_\infty\|_{C^{2,\alpha}(M;\mathbb{R}^n)} < \sigma,$$

Topping [87, Lemma 1] proved a Łojasiewicz-type gradient inequality for maps, $f : S^2 \to S^2$, with small energy, with the latter criterion replacing the usual small $C^{2,\alpha}$ norm criterion of Simon.

Kwon [52, Theorem 4.2] obtains a a Łojasiewicz-type gradient inequality for maps, $f : S^2 \to N$, that are $W^{2,p}(S^2; f^*TN)$-close to a harmonic map, with $1 < p \leq 2$.

Thank you for your attention!

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