Spectral Curves and Ordinary Differential Operators

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Łojasiewicz100, Kraków, Poland, 2026

New results: joint work with J. J. Morales-Ruiz (UPM) and J. P. Ramis (IMT, Toulouse), arXiv:2608.25227

Partially supported by the grant PID2021-124473NB-I00, "Algorithmic Differential Algebra and Integrability" (ADAI) from the Spanish MICINN



Plan of the talk

1. **Commuting ODOs and spectral curves:** the Burchnall–Chaundy theorem [BC23, BC28] and the algebro-geometric integration of Korteweg–de Vries equation (KdV).
2. **Beyond plane curves:** non-planar spectral curves of third-order operators [RZ24] and effective tools to compute them [JPR26].
3. **Differential Galois theory:** spectral Picard–Vessiot fields for algebro-geometric Schrödinger operators [MRZ21]; the case of real coefficients [CH11, CHvdP16].
4. **New result:** the variational equation of the $(1+1)$-dimensional KdV equation around cnoidal waves **[MRZ26]**.

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2. Beyond plane curves: third-order operators
3. Differential Galois theory of spectral problems
4. New result: the KdV variational equation
5. Conclusions

Commuting ordinary differential operators

Spectral Curves and ODOs
M. A. Zurro
Spectral curves
Non-planar curves
Differential Galois theory
New result
Conclusions

Let $\partial = d/dx$ and K a differential field (e.g. of meromorphic functions).

$$L = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_0, \qquad A = \partial^m + v_{m-2}\partial^{m-2} + \cdots + v_0.$$

Burchnall–Chaundy theorem [BC23, BC28]

If $[L, A] = 0$, there is a nonzero polynomial $f \in \mathbb{C}[E, \mu]$ with

$$f(L, A) = 0.$$

The affine curve $\Gamma : \ f(E, \mu) = 0$ is the **spectral curve** of the pair (L, A).

Computing the spectral curve

- Effectively, f can be computed as the **differential resultant**. First proved by E. Previato for analytic coefficients, an in [MRZ20] for any differential field.

$$f(E,\mu) = \partial\mathrm{Res}(L-E, A-\mu)$$

Example

Given $P = a_2\partial^2 + a_1\partial + a_0$ *and* $Q = b_3\partial^3 + b_2\partial^2 + b_1\partial + b_0$ *in* $K[\partial]$,

$$S_0(P,Q) = \begin{bmatrix} a_2 & a_1 + 2\partial(a_2) & a_0 + 2\partial(a_1) + \partial^2(a_2) & 2\partial(a_0) + \partial^2(a_1) & \partial^2(a_0) \\ 0 & a_2 & a_1 + \partial(a_2) & a_0 + \partial(a_1) & \partial(a_0) \\ 0 & 0 & a_2 & a_1 & a_0 \\ b_3 & b_2 + \partial(b_3) & b_1 + \partial(b_2) & b_0 + \partial(b_1) & \partial(b_0) \\ 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}.$$

Common eigenvalues

At the generic point of Γ we obtain a common right factor:

$$L - E = N_1(\partial - \sigma) \quad , \quad A - \mu = N_2(\partial - \sigma)$$

with $\partial - \sigma$ the first subresultant (by work of M. Chardin (1991). Differential resultants and subresultants.)

Points of $\Gamma \longleftrightarrow$ Common eigenfunction:

$$(L - E)\varphi = 0 \ , \ (A - \mu)\varphi = 0$$

Coupled spectral problem!!!

The KdV equation and its Lax pair

The *Korteweg–de Vries* equation, in $(1+1)$ dimensions,

$$u_t = u_{xxx} - 6uu_x, \text{ with } u = u(x,t), \tag{1}$$

D. J. Korteweg and G. de Vries, *Philos. Mag.* **39** (1895), 422–443.

- Lax representation of KdV:
  $L_t = [L, A]$, with $L = -\partial^2 + u$ and $A = -4\partial^3 + 6u\partial + 3u_x$.
- Stationary solutions $(u_t = 0) \Longleftrightarrow [A, L] = 0 \Longrightarrow$ a spectral curve.

Historical context

1. Burchnall–Chaundy (1923, 1928): commuting operators satisfy a polynomial relation, which defines the spectral curve.
2. Novikov (1974): periodic KdV problem via stationary operators of the hierarchy (the "Novikov equations"), opening the door to finite-gap solutions.
3. Dubrovin (1975): explicit reconstruction of those solutions via theta functions on the spectral curve, viewed as a Riemann surface.
4. Krichever (1977): algebro-geometric integration of integrable hierarchies in general, using Baker–Akhiezer functions on the curve.

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Centralizers and non-planar spectral curves [RZ24]

- For an **ODO L of order n**, its centralizer $\mathcal{Z}(L)$ is a free $\mathbb{C}[L]$-module whose rank divides n.
  (by the work of Goodearl, K. R. (1983). Centralizers in differential, pseudo-differential, and fractional differential operator rings.)
- **Centralizers:** effective classification of centralizers of ODOs by differential-algebraic methods [Rue26].
- For a **third-order** operator M with centralizer of rank 3:

$$\mathcal{Z}(M) = \mathbb{C}[M, A_1, A_2] \simeq \mathbb{C}[\Gamma] = \frac{\mathbb{C}[\nu, \mu_1, \mu_2]}{BC(M)},$$

  where $BC(M)$ is the **Burchnall–Chaundy ideal**.

Effective computation

- Generators of $BC(M)$ from differential resultants:

$$f_i = \partial\mathrm{Res}(M-\nu, A_i-\mu_i),\ i=1,2, \qquad f_3^r = \partial\mathrm{Res}(A_1-\mu_1, A_2-\mu_2).$$

- **Burchnall–Chaundy ideals:** Gröbner bases of $BC(M)$ [JPR26]; primality and explicit generators.
- $\Gamma \subset \mathbb{C}^3$ is in general a **non-planar** spectral curve: no single pair of commuting operators captures it. In [RZ24], a spatial spectral curve is calculated explicitly for the first time.

Message

Computational differential algebra does not merely assist in the proofs — it *produces* the geometric objects: the curves themselves.

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The coupled spectral problem

For u a stationary KdV potential, consider the simultaneous eigenvalue problem

$$L\varphi = E\varphi \quad , \quad A\varphi = \mu\varphi \ .$$

- Differential Galois theory:
  Picard–Vessiot theory is the algebraic framework for linear differential equations.
- The spectral parameter runs over a curve.

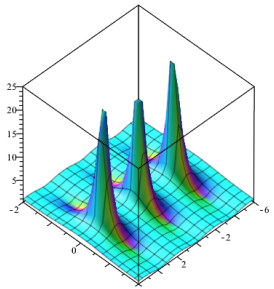
Spectral Picard–Vessiot fields [MRZ21]

The notion of **spectral Picard–Vessiot extension** links the geometry of the spectral curve to the differential Galois structure of the eigenvalue problem: the curve acts as a moduli space of Picard–Vessiot extensions of L.

Running example: the cnoidal wave

- A *cnoidal wave* is a periodic, nonlinear, exact wave solution of KdV, expressed in terms of the Jacobi function cn — equivalently, of the Weierstrass function $\wp$.
- Moving frame $z = x + ct$, $c \neq 0$: $u_t = u_{zzz} - 6uu_z - cu_z$, with stationary solution

$$u_0(z) = 2\wp(z; g_2, g_3) - \frac{c}{6}.$$



$\wp$ with $g_2 = 1$, $g_3 = 0$: white = pole, black = zero.

Now $\partial = \partial_z$,

$$L = -\partial^2 + u_0 \text{ (Lamé–Schrödinger)}, \qquad A_3 = -4\partial^3 + (6u_0 + c)\partial + 3u_{0,z},$$

with Lax equation $L_t = [L, A_3]$; for $u = u_0$, $[L, A_3] = 0$.

The spectral curve Γ_0 [MRZ21]

Since $[L, A_3] = 0$,

$$\Gamma_0 : \quad \mu^2 = -16\widetilde{E}^3 + 4g_2\widetilde{E} - 4g_3 \quad , \quad \widetilde{E} := E + \frac{c}{6}.$$

- Uniformization: $\widetilde{E}(s) = -\wp(s)$, $\mu(s) = 2\wp'(s)$ (same lattice); the involution $\iota : (\widetilde{E}, \mu) \mapsto (\widetilde{E}, -\mu)$ is $s \mapsto -s$, and the branch points $B_i = (-e_i, 0)$ are its fixed points.
- Factorization of $L - \widetilde{E}$ over Γ_0 via $\widetilde{\sigma}(z, s)$, computed with differential subresultants,

$$(-\partial - \widetilde{\sigma})(\partial - \widetilde{\sigma}),$$

Hermite–Halphen eigenfunctions φ_1, φ_2, a basis defined outside the ramification locus $Z = \{B_1, B_2, B_3\}$.

$$\varphi_i(z,s) = \frac{\sigma\big(z + (-1)^{i+1}s\big)}{\sigma(s)\sigma(z)}\, e^{(-1)^i z\zeta(s)}, \qquad \text{in } \Gamma_0 \setminus Z,$$

Eigenvector bundle over Γ_0

Spectral Curves and ODOs

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Spectral curves

Non-planar curves

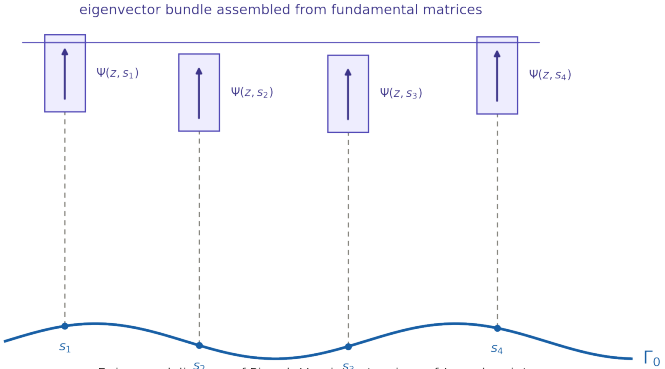
Differential Galois theory

New result

Conclusions

$E \longrightarrow \Gamma_0$

total space (bundle) = fiber at each $s \in \Gamma_0$



Γ_0 is a moduli space of Picard--Vessiot extensions of L: each point s corresponds to the extension generated by the fundamental matrix $\Psi(z, s)$.

The curve Γ_0 for $g_2 = 1$ and $g_3 = 0$

$$\Gamma_0 : \quad \mu^2 = -16\widetilde{E}^3 + 4\widetilde{E}.$$

The branch points $\Lambda_1 = -1/2,\ \Lambda_2 = 0,\ \Lambda_3 = 1/2$ are the **real** roots of the cubic.

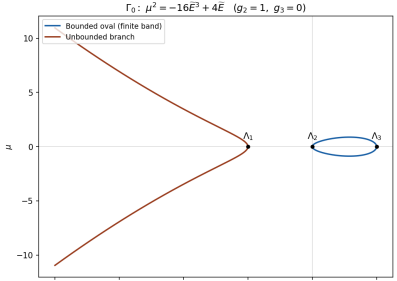


Figure: Real points of Γ_0

On Real coefficients

- For operators with sufficiently regular **real** coefficients, the differential Galois theory of spectral problems remains largely unexplored.

Pioneering work [CH11, CHvdP16]

Crespo, Hajto and van der Put developed a Picard–Vessiot theory over **real** differential fields (real Picard–Vessiot fields), and Crespo and Hajto related Picard–Vessiot theory to the Jacobian problem.

- **Question:** how do the real structures of the spectral curve and of the Picard–Vessiot fields interact?

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New result: overview

J. J. Morales-Ruiz, J. P. Ramis, M. A. Zurro, *From Spectral Curves to Variational Equations: Geometry and Differential Galois Theory of KdV Cnoidal Waves*, arXiv:2608.25227 (2026); revised version in preparation.

Guiding question: what spectral-geometric structure underlies the *variational equation* of KdV around a cnoidal wave? (Stability, perturbation theory, Ziglin–Morales-Ruiz–Ramis criteria [MR01].)

arXiv:2608.25227

Answer arXiv:2608.25227

Two **interacting spectral curves**: Γ_0 of the Lamé–Schrödinger operator, and a **non-planar** curve Γ_1 of the third-order variational operator.

The variational equation

In the moving frame $z = x + ct$ the KdV equation reads $u_t = F[u] := u_{zzz} - 6uu_z - cu_z$. Its Gateaux derivative at u_0 gives

$$\xi_t = -M\xi, \qquad M = -\partial^3 + (6u_0 + c)\partial + 6u_{0,z} = \partial L_0, \qquad L_0 := -\partial^2 + 6u_0 + c.$$

Separating variables, $\xi = e^{-\nu t}\eta(z)$, leads to the spectral problem

$$M\eta = \nu\eta,$$

for a **third-order** operator: exactly the setting of [RZ24].

Around the cnoidal wave

For $u_0(z) = 2\wp(z) - \frac{c}{6}$:

- **Variational equation:** $\xi_t = -M\xi, \quad M = -\partial^3 + 12\wp\,\partial + 12\wp' = \partial L_0$, $L_0 = -\partial^2 + 12\wp$ (Lamé, $n = 3$).
- **Adjoint variational equation:** $\psi_t = M^\dagger\psi, \quad M^\dagger = -L_0\partial = \partial^3 - 12\wp\,\partial$ (formal adjoint).
- If ψ solves the adjoint equation, then $\xi = \partial_z\psi$ solves the variational equation, since $\partial M^\dagger = -\partial L_0\partial = -M\partial$.
- To solve the adjoint equation we use the Hermite–Halphen basis over Γ_0 and the **second symmetric power** of $L - E$ [BMW97].

Second symmetric power: a bridge to the variational equation

$$(L-E)^{\odot 2} = -\partial^3 + 4(u-E)\partial + 2u_z.$$

If φ solves the Lax system $L\varphi = E\varphi$, $\varphi_t = -A_3\varphi$, then $\psi = \varphi^2$ satisfies

$$-\psi_{zzz} + 4(u-E)\psi_z + 2u_z\psi = 0, \qquad \psi_t + (2u + 4E + c)\psi_z - 2u_z\psi = 0.$$

Proposition arXiv:2608.25227

Eliminating E gives $\psi_t = \psi_{zzz} - (6u + c)\psi_z = M^\dagger\psi$. Hence any closed-form solution of the spectral problem yields a closed-form solution of the adjoint variational equation.

First main result: a unifying mechanism

Theorem A

arXiv:2608.25227

The second symmetric power of $L - \widetilde{E}$ unifies

- *squared eigenfunctions*,
- *conserved densities* of the KdV hierarchy,
- *solutions of the adjoint variational equation*,

realized through **asymptotic expansions in a suitable complete ring**, using a recursion operator of KdV.

Theorem B

arXiv:2608.25227

The variational dynamics admits a spectral representation over a *second* spectral curve Γ_1, associated with the **centralizer of** M — not the classical Schrödinger curve, but one intrinsic to the variational operator itself.

The centralizer of M and the curve Γ_1

$\mathcal{Z}(M) = \mathbb{C}[M, A_1, A_2]$, a free $\mathbb{C}[M]$-module of rank 3, with

$$A_2 = \partial^5 - 20\wp\,\partial^3 - 40\wp'\,\partial^2 + (18g_2 - 120\wp^2)\,\partial + 1,$$
$$A_1 = \partial^7 - 28\wp\,\partial^5 - 84\wp'\,\partial^4 - 560\wp^2\,\partial^3 - 560\wp\wp'\,\partial^2 + (-1120\wp^3 + 952g_2\wp + 160g_3)\,\partial + 784g_2\wp'.$$

Lemma arXiv:2608.25227

$MA_1 = 68\,g_2\,M^2 - (A_2 - 1)^2$.

- $BC(M) = (f_1, f_2, f_3)$ is prime [RZ24, JPR26]; with $m = \mu_2 - 1$,

$$f_2 = \nu^5 + 16g_3\nu^3 - 4g_2\nu^2 m + m^3, \qquad \nu\mu_1 = 68g_2\nu^2 - m^2 \text{ on } \Gamma_1.$$

Γ_1 is an elliptic curve

Theorem arXiv:2608.25227

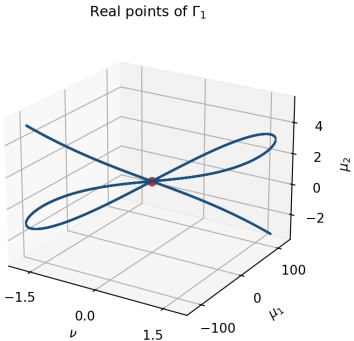
Let $\Delta = g_2^3 - 27g_3^2 \neq 0$ and Λ the period lattice of $\wp$. The map $\Phi : \mathbb{C}/\Lambda \setminus \{0\} \to \Gamma_1,\ b \mapsto (\nu, \mu_1, \mu_2)$,

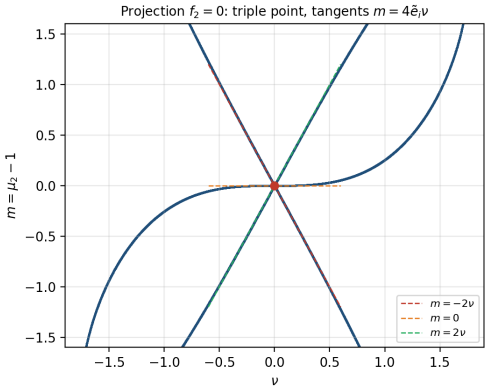
$$\nu = -4\wp'(b), \quad \mu_1 = 16\wp'(b)\big(4\wp(b)^2 - 17g_2\big), \quad \mu_2 = 1 + 16\wp(b)\wp'(b),$$

is the normalization of Γ_1.

- The three half-periods ω_i are sent to the single point $P_0 = (0, 0, 1)$ over $\nu = 0$: an **ordinary triple point**.
- Its tangents are $m = 4\widetilde{e}_i\,\nu$, so its three branches are labelled by the **branch points of** Γ_0.

The curve Γ_1 for $g_2 = 1$ and $g_3 = 0$





Second main result: explicit solutions over Γ_1

Factoring $M - \nu$ using its centralizer yields an intrinsic right factor:

$$M - \nu = -R\,(\partial + \sigma), \qquad \sigma = \frac{N}{D},$$

$$N = -\nu^3 - 4\wp'\nu^2 + (8g_2\wp - 96\wp^3)\nu + (2g_2 - 24\wp^2)\,m,$$

$$D = (4\wp' + \nu)(4\wp\,\nu + m) \;=\; 16\,\nu\,\big(\wp'(z) - \wp'(b)\big)\big(\wp(z) - \wp(b)\big).$$

Theorem C arXiv:2608.25227

For $P = \Phi(b)$, let b_1, b_2 be the other solutions of $\wp'(z) = \wp'(b)$, so that $b + b_1 + b_2 \equiv 0$. Then

$$\xi(z,t) = e^{4\wp'(b)\,t}\; e^{\kappa z}\,\frac{\sigma(z+b)\,\sigma(z-b_1)\,\sigma(z-b_2)}{\sigma(z)^3}, \qquad \kappa = \zeta(b_1) + \zeta(b_2) - \zeta(b),$$

solves the variational equation $\xi_t = -M\xi$. This is a **Hermite–Halphen type formula** for a third-order operator.

Third main result: branch points, where everything coincides

Theorem D arXiv:2608.25227

At the ramification locus $Z = \{B_1, B_2, B_3\}$ of Γ_0:

- Hermite–Halphen eigenfunctions degenerate into classical **Lamé solutions**;
- the squared eigenfunctions give **static** solutions of the adjoint equation in $\langle 1, \wp \rangle$; their derivative is $\wp' = \frac{1}{2} u_{0,z}$, the **unique elliptic static solution** of the variational equation;
- on the three branches of Γ_1 through the triple point, which are labelled by Z, the intrinsic factor tends to $\partial - \wp''/\wp'$ and produces the **same** solution $\wp'$.

The branch points are a **unification locus** for three independent constructions.

A computed example

Case $g_2 = 1$, $g_3 = 0$, $\nu = \frac{1}{10}$. The point $P \in \Gamma_1$ is given by

- $m = \mu_2 - 1 \approx 2.500004 \cdot 10^{-4}$, a root of $m^3 - 4\nu^2 m + \nu^5 = 0$ (Cardano, *casus irreducibilis*);
- $\mu_2 \approx 1.000250000391, \quad \mu_1 = \dfrac{68\nu^2 - m^2}{\nu} \approx 6.799999375$;
- $\wp'(b) = -\frac{1}{40}, \quad \wp(b) = -\dfrac{m}{4\nu} \approx -6.250010 \cdot 10^{-4}$,
  $\wp(b_{1,2}) \approx 0.5003122, \ -0.4996872.$

$$\xi(z,t) = e^{-t/10}\, e^{\kappa z}\, \frac{\sigma(z+b)\,\sigma(z-b_1)\,\sigma(z-b_2)}{\sigma(z)^3}.$$

ξ has a pole of order 3 at the lattice points and simple zeros at $-b$, b_1, b_2; everything has been verified symbolically.

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Conclusions and perspectives

1. **Spectral curves** of commuting ODOs, from Burchnall–Chaundy to **non-planar** curves of third-order operators, are effectively computable (differential resultants, Gröbner bases).
2. **Spectral Picard–Vessiot fields** give the differential Galois counterpart of this geometry.
3. **New [MRZ26]**: for the KdV variational equation around cnoidal waves, two interacting curves Γ_0 and Γ_1 appear. The curve Γ_1 is a "copy" of the elliptic curve of the wave; the solutions are given by **Hermite–Halphen type** formulas.

Perspectives: beyond genus one and further members of the KdV hierarchy; a Galoisian reading of the ramification locus. Picard-Vessiot spectral theory for real differential fields is a challenging problem..

References

[BC23] J. L. Burchnall, T. W. Chaundy, *Commutative ordinary differential operators*, Proc. London Math. Soc. (2) **21** (1923), 420–440.

[BC28] J. L. Burchnall, T. W. Chaundy, *Commutative ordinary differential operators*, Proc. R. Soc. Lond. **118** (1928), 557–583.

[BMW97] M. Bronstein, T. Mulders, J.-A. Weil, *On symmetric powers of differential operators*, Proc. ISSAC '97 (1997).

[CH11] T. Crespo, Z. Hajto, *Picard–Vessiot theory and the Jacobian problem*, Israel J. Math. **186** (2011), 401–406.

[CHvdP16] T. Crespo, Z. Hajto, M. van der Put, *Real and p-adic Picard–Vessiot fields*, Math. Ann. **365** (2016), no. 1, 93–103.

[JPR26] A. Jiménez-Pastor, S. L. Rueda, *Gröbner bases of Burchnall–Chaundy ideals for ordinary differential operators*, arXiv:2602.11402.

[MR01] J. J. Morales-Ruiz, J. P. Ramis, *Galoisian obstructions to integrability of Hamiltonian systems* (2001).

[MRZ20] J. J. Morales-Ruiz, S. L. Rueda, M. A. Zurro, *Factorization of KdV Schrödinger operators using differential subresultants*, Adv. App. Math , (2020), no. 120, 102065.

[MRZ21] J. J. Morales-Ruiz, S. L. Rueda, M. A. Zurro, *Spectral Picard–Vessiot fields for algebro-geometric Schrödinger operators*, Ann. Inst. Fourier **71** (2021), no. 3, 1287–1324.

[MRZ26] J. J. Morales-Ruiz, J. P. Ramis, M. A. Zurro, *From spectral curves to variational equations: geometry and differential Galois theory of KdV cnoidal waves*, arXiv:2608.25227 (2026).

[Rue26] S. L. Rueda, *On the classification of centralizers of ODOs: an effective differential algebra approach* (2026).

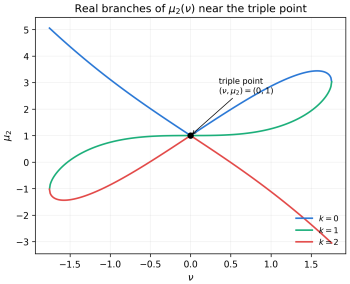
[RZ24] S. L. Rueda, M. A. Zurro, *Spectral curves for third-order ODOs*, Axioms **13** (2024), no. 4, 274.

Thank you.

New results: arXiv:2608.25227

Partially supported by the grant PID2021-124473NB-I00, "Algorithmic Differential Algebra and Integrability" (ADAI), from the Spanish MICINN

Backup: the ordinary triple point of $\{f_2 = 0\}$



- An **ordinary triple point** at $(\nu, \mu_2) = (0, 1)$, with tangents $\mu_2 - 1 = 4\widetilde{e}_i\,\nu$; for $g_2 = 1$, $g_3 = 0$ these are $\mu_2 - 1 = -2\nu,\ 0,\ 2\nu$: three real branches crossing transversely.
- At infinity, $\{f_2 = 0\}$ has an A_4 singularity; hence it has genus $6 - 3 - 2 = 1$.

Backup: static solutions ($\nu = 0$)

Since $M = \partial L_0$ and $M^\dagger = -L_0\partial$:

- Ker $M = \langle\, \wp',\ z\wp' + 2\wp,\ 6\wp^2 + 3\zeta\wp' - g_2 \,\rangle$, with Wronskian $-\Delta$.
- Ker $M^\dagger = \langle\, 1,\ \wp,\ \chi \,\rangle$, where $\chi = 9g_3(z\wp - \zeta) - 6g_2\zeta\wp - 3g_2\wp' + \frac{1}{2}g_2^2 z$.
- $\partial : \mathrm{Ker}\, M^\dagger \to \mathrm{Ker}\, M$ has image Ker L_0, of codimension one.
- The only elliptic static solution is $\wp'$ (by the Legendre relation).
- The eigenvalue $\nu = 0$ is not semisimple: $M(1) = 12\wp'$, so that $\xi = 1 - 12t\wp'$ (the Galilean mode) solves $\xi_t = -M\xi$.

Backup: the Weierstrass functions

Fix a lattice $\Lambda = 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z} \subset \mathbb{C}$, $\Lambda^* = \Lambda \setminus \{0\}$.

- $\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$, with $\wp'^2 = 4\wp^3 - g_2\wp - g_3$.
- $\zeta(z) = \frac{1}{z} + \sum_{w\in\Lambda^*} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right)$, with $\zeta' = -\wp$.
- $\sigma(z) = z \prod_{w\in\Lambda^*} \left(1 - \frac{z}{w}\right) \exp\left(\frac{z}{w} + \frac{1}{2}\left(\frac{z}{w}\right)^2 \right)$, with $\sigma'/\sigma = \zeta$, hence $(\log\sigma)'' = -\wp$.