Solution of the Metric Arnold corank Problem

A. Fernandes, Z. Jelonek, J. E. Sampaio

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The classification of hypersurface singularities stands as a cornerstone of modern singularity theory and catastrophe theory, largely shaped by the foundational works of Vladimir Arnold. In his seminal classification scheme, Arnold organized singularities based on their complexity, introducing the celebrated A_k, D_k, E_k series for simple singularities, followed by unimodal and bimodal families. Central to this structural hierarchy is the concept of the corank of a singularity – defined as the corank of the Hessian matrix of the function germ at the critical point.

The corank serves as a fundamental invariant that dictates the intrinsic dimensionality of the degeneracy. By the Splitting Lemma, any function germ can be decomposed into a non-degenerate quadratic form and a degenerate part whose variables correspond precisely to the corank. Consequently, the corank acts as the primary branching criterion in Arnold's classification: a corank of 1 restricts the singularity to the A_k series, while higher coranks unlock the significantly more intricate geometry of the D_k, E_k, and exceptional families.

While the algebraic properties of these classes are well-understood, decoding their topology and metric geometry remains an active and challenging frontier. A long-standing problem in this direction is understanding how the corank is intrinsically constrained by metric properties of a singularity. In a parallel direction, the metric Zariski multiplicity conjecture posits a similarly rigid relationship between the ambient bi-Lipschitz type of a hypersurface singularity and its multiplicity.

In this lecture, we bridge these foundational directions by establishing two major results. First, we provide a definitive solution to the metric Arnold corank problem. Second, we deliver a complete proof of the metric Zariski multiplicity conjecture for multiplicity two.

More precisely, let $f\colon (\mathbb{C}^n,0)\to(\mathbb{C},0)$ be the germ of a holomorphic function at the origin. Let $(V(f),0)$ be the germ of the zero set of f at the origin. We recall that the **corank** of the germ of a holomorphic function $f\colon (\mathbb{C}^n,0)\to(\mathbb{C},0)$ is the corank of the Hessian matrix f at 0. We denote the corank of f by $\mathrm{crk}(f)$. When f is a reduced function, the **corank** of $(V(f),0)$ is $\mathrm{crk}(V(f),0) := \mathrm{crk}(f)$, and the **rank** of $(V(f),0)$ is $\mathrm{rk}(V(f),0) := n - \mathrm{crk}(f)$.

In 1975, Arnold asked whether the corank of a holomorphic function is an ambient topological invariant:

Problem[Arnold corank problem] Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be germs of reduced holomorphic functions. If there is a homeomorphism $\varphi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\varphi(V(f)) = V(g)$, is it true that $\mathrm{crk}(f) = \mathrm{crk}(g)$?

This question is also stated in the book Arnold's Problems. In this book, there is also a comment from Vassiliev that nothing is known to him about this problem.

From the point of view of the metric geometry, we have the following question.

Problem[Metric Arnold corank problem] Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be germs of reduced holomorphic functions. If there is a bi-Lipschitz homeomorphism $\varphi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\varphi(V(f)) = V(g)$, is it true that $\mathrm{crk}(f) = \mathrm{crk}(g)$?

The corank of a holomorphic function at a singular point measures how far a singularity is from being a Morse singularity (which corresponds to corank zero). This measure is operationally defined by the powerful Splitting Lemma (also known as the Generalized Morse Lemma), which shows that any holomorphic function near a critical point can be decomposed into a quadratic part plus a residual function depending only on the corank variables.

In other words: given a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with $r = n - \mathrm{crk}(f)$, there is a holomorphic diffeomorphism $\psi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that

$$f \circ \psi(z) = z_1^2 + \cdots + z_r^2 + h(z_{r+1}, ..., z_n),$$

where $h : (\mathbb{C}^{n-r}, 0) \to (\mathbb{C}, 0)$ is a holomorphic function such that $\mathrm{ord}_0(h) > 2$.

In this way, the corank determines the dimension of the "essential" singularity space that carries the true complexity of the singularity. Therefore, the Arnold corank problem (resp. the metric Arnold corank problem) asks whether the decomposition given by the Splitting Lemma, which is a priori an analytic invariant, depends only on the embedded topological (resp. bi-Lipschitz) type of the zero sets of the involved functions.

In this lecture, we provide a complete positive answer to the metric Arnold corank problem.

Theorem 1 Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced holomorphic functions which are singular at the origin. Assume that there is a bi-Lipschitz homeomorphism $\phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ that sends the zeros of f onto the zeros of g. Then $\mathrm{crk}(f) = \mathrm{crk}(g)$.

We also provide a positive answer to the Arnold corank problem when $n = 3$.

Theorem 2 Let $f, g\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be two reduced holomorphic functions. Assume that there is a homeomorphism $\phi\colon (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ sending the zeros of f onto the zeros of g. Then $\mathrm{crk}(f) = \mathrm{crk}(g)$.

Another problem addressed here is the following conjecture, which is a metric version of the famous Zariski multiplicity problem:

Conjecture A. *Let $X, Y \subset \mathbb{C}^n$ be two complex analytic hypersurfaces. If their germs at zero are ambient bi-Lipschitz homeomorphic, then their multiplicities $m(X,0)$ and $m(Y,0)$ are equal.*

Despite many efforts this conjecture is still open. We have proved that this conjecture follows from the following metric version of Zariski problem B:

Conjecture B. *Let $C(X), C(Y) \subset \mathbb{C}^n$ be two $(n-1)-$dimensional algebraic cones with bases X, Y. If there is a bi-Lipschitz homeomorphism*

$$\varphi\colon (\mathbb{C}^n, C(X), 0) \to (\mathbb{C}^n, C(Y), 0),$$

then X is homeomorphic to Y.

In this lecture, we answer the above question partially: we show that if two algebraic $(n-1)-$dimensional cones $C(X), C(Y) \subset \mathbb{C}^n$ with isolated singularities are homeomorphic, then they have the same degree and, in particular, X is homeomorphic to Y. We prove that if two algebraic $(n-1)$-dimensional cones $C(X), C(Y) \subset \mathbb{C}^n$ are ambient homeomorphic, then their bases X and Y have the same Euler characteristic.

As a consequence of Theorem 2, we obtain that Conjecture A is true for multiplicity two:

Theorem 3 Let $(X,0),(Y,0) \subset (\mathbb{C}^n,0)$ be germs of analytic hypersurfaces that are ambient bi-Lipschitz equivalent. If $m(X,0) = 2$, then also $m(Y,0) = 2$.

In particular, this implies that Conjecture B is true if the degree of $C(X)$ is 2.

Definition

Let $X \subset \mathbb{CP}^n$ be an algebraic variety. We assume $\mathbb{CP}^n$ to be a hyperplane at infinity of $\mathbb{CP}^{n+1}$. Then by an algebraic cone $\overline{C(X)} \subset \mathbb{CP}^{n+1}$ with base X we mean the set

$$\overline{C(X)} = \bigcup_{x \in X} \overline{O, x},$$

where O is the center of coordinates in $\mathbb{C}^{n+1} \subset \mathbb{CP}^{n+1}$, and $\overline{O, x}$ means the projective line which goes through O and x. By an affine cone $C(X)$ we mean $\overline{C(X)} \setminus X$. By the link of $C(X)$ we mean the set $L = \{x \in C(X) : ||x|| = 1\}$.

Theorem 1

Let $V \subset \mathbb{CP}^{n+1}$ be a smooth algebraic hypersurface. Then the integral (co)homology of V is torsion free, and the corresponding Betti numbers are given as follows:

(1) $b_i(V) = 0$ for $i \neq n$ odd or $i \notin [0, 2n]$.

(2) $b_i(V) = 1$ for $i \neq n$ even and $i \in [0, 2n]$.

(3) $b_n(V) = \frac{(d-1)^{n+2}+(-1)^{n+1}}{d} + \frac{3(-1)^n+1}{2}$.

Theorem 2

Let $P = C(X), R = C(Y) \subset \mathbb{C}^{n+1}$ be two algebraic n dimensional cones with smooth bases X, Y. If P, R are homeomorphic, then $\deg P = \deg R$. In particular, X and Y are also homeomorphic.

Proof

Let deg $P = p$, deg $R = r$. Since theorem is true for $1-$dimensional cones, we can assume that P, R have connected bases. By Prill's Theorem, we can assume that 0 is not topologically regular point of P and R, because otherwise both cones are hyperplanes. Thus, $p, r \geq 2$. Hence $P^* = P \setminus \{0\}$ is homeomorphic to $R^* = R \setminus \{0\}$. Let L_P, L_R be links of P, R.

Hence L_P is a deformation retract of P^*, similarly L_R is a deformation retract of R^*, Since P^* and R^* are homeomorphic we have that L_P is homotopy equivalent to L_R. Let X, Y be bases of P, R. We have the Hopf fibration $\pi : L_P \to X, \pi' : L_R \to Y$ whose fibers are circles. Analizing the spectral sequences of the mappings π and π' , we have that the corresponding Betti numbers are given as follows:

1) $b_i(L_P, \mathbb{Q}) = b_i(X, \mathbb{Q}) - b_{i-2}(X, \mathbb{Q})$ if $i \leq \dim X$
2) $b_{i+1}(L_P, \mathbb{Q}) = b_i(X, \mathbb{Q}) - b_{i+2}(X, \mathbb{Q})$ if $i \geq \dim X$.
3) $b_i(L_R, \mathbb{Q}) = b_i(Y, \mathbb{Q}) - b_{i-2}(Y, \mathbb{Q})$ if $i \leq \dim X$
4) $b_{i+1}(L_R, \mathbb{Q}) = b_i(Y, \mathbb{Q}) - b_{i+2}(Y, \mathbb{Q})$ if $i \geq \dim X$.

Since P, R have homotopic links $L_P \sim L_R$, we see by 1) - 4) and Theorem 1 that $b_n(X, \mathbb{Q}) = b_n(Y, \mathbb{Q})$, i.e.,

$$\frac{(p-1)^{n+1} + (-1)^n}{p} = \frac{(r-1)^{n+1} + (-1)^n}{r}.$$

Since the function $f(x) = \frac{(x-1)^{n+1}+(-1)^n}{x}$ increases for $x \geq 2$, then $p = r$.

We see that homeomorphic cones of dimension n in $\mathbb{C}^{n+1}$ with smooth bases have homeomorphic bases. If bases are non-smooth the problem whether bases are homeomorphic is a difficult open problem which implies metric version of Zariski multiplicity Conjecture. However in the general case we can prove the following result which slightly confirms a positive answer to this problem:

Theorem 3

Let $P = C(X), Q = C(Y) \subset \mathbb{C}^{n+1}$ be n dimensional algebraic cones, which are ambient homeomorphic. Then $\chi(X) = \chi(Y)$.

Proof

Let $f = 0, g = 0$ be reduced equation of cones P and Q respectively and let $d_f = mult_0 P, d_g = mult_0 g$. The global fibration given by f, g are equivalent to local. Hence to compute the Lefschetz number of a monodromy h given by fibration $f : \mathbb{C}^{n+1} \setminus P \to \mathbb{C}^*$ we can assume that the monodromy h is a geometric i.e. $h_f(z) = e^{(2\pi i/d_f)} z$. We start with the following lemma which follows from Theorem 3.3 and Remark 3.4 in Le paper:

Lemma 1

Let $f, g : \mathbb{C}^{n+1} \to \mathbb{C}$ be two reduced holomorphic functions. If germs $(V(f), 0)$ and $(V(g), 0)$ are ambient homeomorphic, then the induced maps in homology of the monodromies of f and g at 0 are conjugated. In particular, the monodromies of f and g at 0 have the same Lefschetz number and $\chi(F_f) = \chi(F_g)$.

Hence for $0 < k < d_f$, we have $\Lambda(h_f^k) = 0$, where $\Lambda(h_f^k)$ denotes the Lefschetz number of h_f^k. Similarly for $0 < k < d_g$, we have $\Lambda(h_f^k) = 0$. It follows from Lemma 1 that $\Lambda(h_f^k) = \Lambda(h_g^k)$ for all natural k. We have $\Lambda(h_f^{d_f}) = \chi(F_f)$ and $\Lambda(h_g^{d_g}) = \chi(F_g)$. Hence if $\chi(F_f) = \chi(F_g) \neq 0$ we have $d_f = d_g$. Now consider a natural mapping $F_f \ni x \to [x] \in \mathbb{P}^n \setminus X$, which is a covering of degree d_f. Hence $\chi(F_f) = d_f \chi(\mathbb{P}^n \setminus X)$. Similarly $\chi(F_g) = d_g \chi(\mathbb{P}^n \setminus Y)$.

This means by Lemma 1 that $\chi(\mathbb{P}^n \setminus X) = \chi(\mathbb{P}^n \setminus Y)$. But $\chi(\mathbb{P}^n \setminus X) = n + 1 - \chi(X)$, $\chi(\mathbb{P}^n \setminus Y) = n + 1 - \chi(Y)$, hence $\chi(X) = \chi(Y)$. If $\chi(F_f) = 0$, then we have by the same argument $\chi(\mathbb{P}^n \setminus X) = \chi(\mathbb{P}^n \setminus Y) = 0$ and $\chi(X) = \chi(Y) = n + 1$.

Main Theorem

Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be two reduced holomorphic functions that are singular at the origin. Assume that there is a bi-Lipschitz homeomorphism* $\phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *that sends the zeros of* f *onto the zeros of* g*. Then* $\operatorname{crk}(f) = \operatorname{crk}(g)$.

We divide our proof into two lemmas. The first lemma states, in particular, that the theorem holds true if $m(V(f),0) = m(V(g),0) = 2$.

Lemma 1

Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be two reduced holomorphic functions. Assume that there is a bi-Lipschitz homeomorphism* $\phi\colon (V(f), 0) \to (V(g), 0)$. *If* $m(V(f), 0) = m(V(g), 0) = 2$, *then* $\operatorname{crk}(f) = \operatorname{crk}(g)$.

Proof

Let $r = n - \mathrm{crk}(f)$ and $s = n - \mathrm{crk}(g)$. Then $\mathrm{crk}(f) = \mathrm{crk}(g)$ if and only if $r = s$. Since $m(V(f), 0) = m(V(g), 0) = 2$, we have that r and s are positive. After a change of coordinates, if necessary, we may assume that f and g are written in the following way:

$$f(z_1, ..., z_r, z_{r+1}, ..., z_n) = z_1^2 + ... + z_r^2 + h(z_1, ..., z_n)$$

and

$$g(z_1, ..., z_s, z_{s+1}, ..., z_n) = z_1^2 + ... + z_s^2 + \tilde{h}(z_1, ..., z_n)$$

where $h, \tilde{h}\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are holomorphic functions such that $\mathrm{ord}_0 h > 2$ and $\mathrm{ord}_0 \tilde{h} > 2$.

The case $r \leq 2$ was done earlier. So, by symmetry, we may assume that $r, s \geq 3$.
Let S be a tangent cone to $V(f)$ at 0, and T be a tangent cone to $V(g)$ at 0. Hence, $S = \{z \in \mathbb{C}^n : z_1^2 + ... + z_r^2 = 0\}$ and $T = \{z \in \mathbb{C}^n : z_1^2 + ... + z_s^2 = 0\}$. In particular, $\mathrm{Sing}(S) = \{0\} \times \mathbb{C}^{n-r}$ and $\mathrm{Sing}(T) = \{0\} \times \mathbb{C}^{n-s}$.

There is a bi-Lipschitz homeomorphism $h\colon S \to T$. By the Sampaio Theorem, $h(\mathrm{Sing}(S)) = \mathrm{Sing}(T)$. Then, $\{0\} \times \mathbb{C}^{n-r}$ and $\{0\} \times \mathbb{C}^{n-s}$ are bi-Lipschitz homeomorphic, and therefore $r = s$.

Hence it is enough to prove the following:

Lemma 2

Let S be an irreducible algebraic cone of degree two. Let T be another algebraic cone. Assume that there is bi-Lipschitz mapping $\phi : S \to T$. Then $\deg\ T = 2$. *Moreover, we can find a linear isomorphism $L : S \to T$.*

Proof

We can assume that $S = \{z \in \mathbb{C}^n : z_1^2 + ... + z_r^2 = 0\}, 3 \leq r \leq n$. Assume that our Lemma is false. Let $d > 2$ be a minimal degree of an algebraic cone T such that there is a bi-Lipschitz mapping $\phi : S \to T$. Since smoothness is a bi-Lipschitz invariant and T is bi-Lipschitz equivalent to S we have that also $Sing(T)$ in some system of coordinates is equal to $\mathbb{C}^{n-r} \times \{0\}$. Additionally we can choose this system of coordinates so generic, that $T' = T \cap \{0\} \times \mathbb{C}^r$ is a cone with isolated singularity.

Now consider two points $z, w \in Sing(S)$. Hence $z, w \in \mathbb{C}^{n-r} \times \{0\}$. There is a translation $S : \mathbb{C}^n \ni x \mapsto x + w - z \in \mathbb{C}^n$, which is of course bi-Lipschitz and which preserves S. Hence $(\mathbb{C}^n, S, z)$ is a bi-Lipschitz equivalent to $(\mathbb{C}^n, S, w)$. Moreover, there is a parametric family of such bi-Lipschitz mappings. Since $(\mathbb{C}^n, S)$ is bi-Lipschitz equivalent to $(\mathbb{C}^n, T)$ we have that the same is true for T. Indeed, we have a parametric family of bi-Lipschitz mappings $\psi_v = \phi T_v \phi^{-1} : \mathbb{C}^n \to \mathbb{C}^n$, where $v \in \mathbb{C}^n$ and $T_v : \mathbb{C}^n \ni x \mapsto x + v \in \mathbb{C}^n$, which preserves T.

In particular if $z \in Sing(T)$ then we have the bi-Lipschitz mapping $d\psi_{\phi^{-1}(z)} \circ \phi : (\mathbb{C}^n, T_0S = S) \to (\mathbb{C}^n, T_zT)$ ($d\psi_{\phi^{-1}(z)}$ denotes here the Sampaio derivation). By our assumption we have either deg $T_z =$ deg T or deg $T_z = 2$. If deg $T_z =$ deg T on the dense subset of $Sing(T)$, then we have that for every point $x \in Sing(T)$ we have $mult_x T = deg\ T$, because $mult$ is lower semicontinous.

Let $\overline{T}$ be a projective closure of T. Since *mult* is lower semicontinous we have that $mult_z \overline{T} = \deg T$ also for $z \in \overline{Sing(T)}$. Let hyperplane W be a part at infinity of $\overline{Sing(T)}$. Take a point $w \in W, a \in T'$ and consider a line $l = \overline{(w, a)}$. By the Bezout Theorem we have $l \subset T$. This means that the projection $\pi : T \to T'$ along $\mathbb{C}^{n-r} \times \{0\}$ has $\mathbb{C}^{n-r} \times \{a\}, a \in T'$ as fibers, i.e., $T = \mathbb{C}^{n-r} \times T'$. This means that S'^* and T'^* have the same homotopy type. Thus as in Theorem 2 we have $\deg S = \deg T$ - a contradiction.

Hence we can assume that on the dense subset of $Sing(T)$ we have $mult = 2$. Take $z \in Sing(T) = \mathbb{C}^{n-r} \times \{0\}$. Since $T_z T$ is quadratic and bi-Lipschitz equivalent with T and $\mathbb{C}^{n-r} \times \{0\} \subset T_z T := T_z$ we have $Sing(T_z) = \mathbb{C}^{n-r} \times \{0\}$. In particular for $x \in \mathbb{C}^{n-r} \times \{0\}$ we have $mult_x T_z = 2$. Since $mult$ is lower semicontinous we have that $mult_z \overline{T_z} = 2$ also for $z \in \overline{Sing(T_z)}$. Let $T'_z = T_z \cap \mathbb{C}^{n-r} \times \{0\}$. Thus the projection $\pi : T_z \to T'_z$ along $\mathbb{C}^{n-r} \times \{0\}$ has $\mathbb{C}^{n-r} \times \{a\}, a \in T'_z$ as fibers, i.e., $T_z = \mathbb{C}^{n-r} \times T'_z$.

Denote coordinates in $\mathbb{C}^{n-r} \times \{0\}$ by x and coordinates in $\{0\} \times \mathbb{C}^r$ by t. Hence T_z has equation

$$\Sigma_{i,j=1}^{r} \alpha_{ij}(x) t_i t_j = 0,$$

where $z = (x, 0)$. Let $A = det[\alpha_{ij}(x)]$. Note that A is a non-zero polynomial on a dense subset of $\mathbb{C}^{n-r} \times \{0\}$, because all tangent cones of degree two have the same rank $r = \operatorname{rank} S$. Moreover, $mult\ T_z = 2$ on $\mathbb{C}^{n-r} \setminus \{A = 0\}$ (here we identify $\mathbb{C}^{n-r} \times \{0\}$ with $\mathbb{C}^{n-r}$) and all α_{ij} vanish on $A = 0$, because the tangent cone if it is quadratic has to have the same rank as S. Let $A = \prod_{j=1}^{s} a_j^{k_j}$ be a decomposition of A into irreducible factors.

Fix i, j and consider the polynomial $\alpha_{i,j}(x) := \alpha$. Thus

$$\alpha := b(x) \prod_{j=1}^{s} a_j^{r_j},$$

where $GCD(b, A) = 1$. We show first that b is a constant. Let f be a homogenous, reduced polynomial which describes T. We know that $\frac{\partial^2 f}{\partial t_i \partial t_j}(x, 0) = 2\alpha$. We can choose so generic system of coordinates that $\frac{\partial a_1}{\partial x_1}(x, 0) \neq 0$. Hence on $\{a_1 = 0\}$ we have

$$\frac{\partial^{r_1} f}{\partial t_i \partial t_j \partial^{r_1} x_1}(x, 0) = Cb(x) \prod_{i=2}^{s} a_i^{r_i} (\frac{\partial a_1}{\partial x_1}(x, 0))^{r_j},$$

where C is a non-zero constant.

Take a point $x \in \{a_1 = 0\}$ in this way that $b(x) \neq 0$ and $a_i(x) \neq 0$ for $i > 1$.Then deg $T_{(x,0)}T$ has degree at most $d - (\deg\ b + \Sigma_{i=2}^{s} \deg\ r_i a_i) < d$. This is a contradiction. Hence b is a constant. In a similar way we can prove that $s = 1$. Finally we show that a_1 is a linear polynomial. Indeed, we have on $\{a_1 = 0\}$:

$$\frac{\partial^{r_1+2} f}{\partial t_i \partial t_j \partial^{r_1} x_1}(x,0) = C(\frac{\partial a_1}{\partial x_1}(x,0))^{r_1},$$

where C is a non-zero constant. Take a point $x \in \{a_1 = 0\}$ in this way that $\frac{\partial a_1}{\partial x_1}(x,0) \neq 0$. Then deg $T_{(x,0)}T$ has degree at most $2 + r_1$. If a_1 is not a linear polynomial, then $2 + r_1 < d$. This is a contradiction. We can assume that $a_1 = x_1$. Thus

$$\frac{\partial^2 f}{\partial t_i \partial t_j}(x,0) = x_1^{d-2} c_{ij},$$

where c_{ij} are constants.

Now consider $\frac{\partial^3 f}{\partial t_i \partial t_j \partial t_k}(x,0) = \beta$. If $d > 3$ we have again $\beta = 0$ on $x_1 = 0$, because otherwise on $\{x_1 = 0\} \subset Sing(T)$ we can find a tangent cone to T of degree less than d. Similarly as above we see that $\beta = Cx_1^{d-3}$ where C is a constant. We can continue in this manner to obtain from the Taylor formula:

$$f(x,t) = \Sigma_{k=2}^{d} 1/k! \Sigma_{i_1,i_2,\ldots,i_k} x_1^{d-k} c_{i_1,i_2,\ldots,i_k} t_{i_1} t_{i_2} \ldots t_{i_k}.$$

On the other hand we know that T is bi-Lipschitz equivalent with S, in particular by Theorem 3 the bases of these two cones have the same Euler characteristic. This implies that also the bases of cones

$$S' = \{(x,t) \in \mathbb{C} \times \mathbb{C}^r : t_1^2 + ... + t_r^2 = 0\}$$

and

$$T' = \{(x,t) \in \mathbb{C} \times \mathbb{C}^r : \Sigma_{k=2}^{d} 1/k! \Sigma_{i_1,i_2,...,i_k} x^{d-k} c_{i_1,i_2,...,i_k} t_{i_1} t_{i_2} ... t_{i_k} = 0\}$$

have the same Euler characteristic.

ndeed, let B, C be bases of the cones S', T' respectively. Consider a new cone $\mathbb{C} \times S'$ with base B'. Then B' in $\mathbb{P}^{r+1}$ is a cone over B with a vertex $A = (1 : 0 : ... : 0)$, i.e., it is a line bundle over B plus one point. Hence, $\chi(B') = \chi(B) + 1$. By the same reason, the base C' the cone $\mathbb{C} \times T'$ has Euler characteristic $\chi(C') = \chi(C) + 1$. Thus if the characteristics of new bases coincide, then also characteristics of old bases coincide. The result is obtained by induction.

Observe that bases of S' and T' have only one singular point $(1:0:0:....:0)$. Moreover the Milnor number at this point is the same at S' and T' and it is equal to one, because both of these singularities are quadratic singularities, i.e., they can be described as $\{z \in \mathbb{C}^r : z_1^2 + ... + z_r^2 + \textit{higher terms in } z = 0\}$. Now we use the following theorem, which is an easy consequence of the Parusiński result:

Theorem 6

Let $X \subset \mathbb{CP}^n$ be of degree d complex projective hypersurface. Assume that X has only isolated singular points $x_1, ..., x_k$ and the Milnor number of X at x_i is equal μ_i. Then the Euler characteristic of X is given by the formula:

$$\chi(X) = (n+1) - \frac{1}{d}(1 + (-1)^n(d-1)^{n+1}) + (-1)^n \Sigma_{i=1}^{k} \mu_i.$$

By Theorem 6 we have
$r+1-\frac{1}{d}(1+(-1)^r(d-1)^{r+1})\pm 1 = r+1-\frac{1}{2}(1+(-1)^r(2-1)^{r+1})\pm 1.$
Since the function $f(x) = \frac{(x-1)^{r+1}+(-1)^r}{x}$ increases for $x \geq 2$, we get $d = 2$, which is a contradiction.
Hence deg $T = 2$.

Moreover, the quadratic functions which describes S and T have the same rank equal to $n - \dim Sing(S) = n - \dim Sing(T)$. This means that they are linearly equivalent, hence there is a linear isomorphism $L : \mathbb{C}^n \to \mathbb{C}^n$, such that $L(S) = T$.

As a direct consequence, we obtain that the multiplicity two is an ambient bi-Lipschitz invariant.

Corollary

Let $(X,0),(Y,0) \subset (\mathbb{C}^n,0)$ be germs of analytic hypersurfaces that are ambient bi-Lipschitz equivalent. If $m(X,0) = 2$, then also $m(Y,0) = 2$.

Remark

In theorem 4 it is important, that X, Y are hypersurfaces. Indeed, we can construct a quadratic three dimensional cone $X \subset \mathbb{C}^7$, which is ambient bi-Lipschitz equivalent to a non quadratic three dimensional cone $Y \subset \mathbb{C}^7$.

In this Section, we prove that the Arnold corank problem has a positive answer for functions of three variables. Let us state some definitions and results that we need for our proof.

Definition

A holomorphic function $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ *is said a* **generalized semi-Brieskorn-Pham** *if there are a holomorphic diffeomorphism* $\phi:(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$*, a holomorphic function* $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$*, and positive integer numbers* $k, a_1, ..., a_{k-1}$ *and* a_k *such that*

$a_1 \leq a_2 \leq .... \leq a_k$; $\operatorname{ord}_0 h > a_k$;

$f\circ\phi(z_1,...,z_n) = z_1^{a_1} + ... + z_k^{a_k} + h(z_1,...,z_n)$ *for all* $z=(z_1,...,z_n)$ *close to* 0.

In this case, if $a_1 = 1$*, we define* $m_1(f) = 1$ *and if* $a_1 > 1$*, we define* $m_1(f) := \max\{i; a_i = a_1\}$.

The following result was proved separately by A'Campo and independently by Lê :

Theorem (A'Campo-Lê's Theorem). *Let* $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be two complex analytic functions such that there is a homeomorphism* $\varphi\colon (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$. *If* $m(V(f), 0) = 1$ *then* $m(V(g), 0) = 1$ *as well.*

The next result was proved by A'Campo.

Theorem *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be a complex analytic function. Then*

$$m(V(f), 0) \leq \inf\{k > 0; \Lambda(h_f^k) \neq 0\},$$

where $h_f : F_f \to F_f$ *is the monodromy of the Milnor fibration of* f*, and* $\Lambda(h_f^k)$ *denotes the Lefschetz number of* h_f^k*. Moreover, if* $\chi(\mathbb{CP}^{n-1} \setminus PV(in(f))) \neq 0$*, then*

$$m(V(f), 0) = \inf\{k > 0; \Lambda(h_f^k) \neq 0\},$$

where $PV(in(f)) = p(V(in(f)) \setminus \{0\})$ *and* p *is the natural projection* $p : \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$*.*

Theorem

Let $f, g\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ *be two reduced holomorphic functions. Assume that there is a homeomorphism* $\phi\colon (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ *such that* $\phi(V(f)) = V(g)$. *Then* $\operatorname{crk}(f) = \operatorname{crk}(g)$.

Proof

Since $\mathrm{rk}(V(f),0) = 3 - \mathrm{crk}(f)$ and $\mathrm{rk}(V(g),0) = 3 - \mathrm{crk}(g)$, we obtain that $\mathrm{crk}(f) = \mathrm{crk}(g)$ if and only if $\mathrm{rk}(V(f),0) = \mathrm{rk}(V(g),0)$. So, we are going to show $\mathrm{rk}(V(f),0) = \mathrm{rk}(V(g),0)$.

We first consider the case that $\mathrm{rk}(V(f),0) \neq 0$ and $\mathrm{rk}(V(g),0) \neq 0$.

Claim 1 If $\mathrm{rk}(V(f),0) \neq 0$ and $\mathrm{rk}(V(g),0) \neq 0$, then $\mathrm{rk}(V(f),0) = \mathrm{rk}(V(g),0)$.

Proof of Claim 1

Assume that $\mathrm{rk}(V(f),0) > 0$ and $\mathrm{rk}(V(g),0) > 0$.
Suppose that $\mathrm{rk}(V(f),0) = 3$. Then, after a change of coordinates, if necessary, we may assume that $f(z_1,z_2,z_3) = z_1^2 + z_2^2 + z_3^2$. In particular, $\mu(f) = 1$. Then, $\mu(g) = 1$. Therefore, $\mathrm{rk}(V(g),0) = 3$ as well.

Now, assume that $\mathrm{rk}(V(f),0)=2$. Then, by the Splitting Lemma, after a change of coordinates, if necessary, we may assume that $f(z_1,z_2,z_3)=z_1^2+z_2^2+az_3^{k+1}$ for some integer $k\geq 2$ and some constant $a\in\mathbb{C}$. Indeed, we may assume that $a\in\{0,1\}$.
If $a=1$, then the zeros of f, $V(f)$, is an A_k singularity, and its link L_f has finite fundamental group. Let $G=\pi(L_f)$. Then $(V(f),0)$ is isomorphic to $(\mathbb{C}^2/G,0)$. Since $\pi_1(L_g)\cong G$, where L_g is the link of the zeros of g, we have that $(V(g),0)$ is isomorphic to $(\mathbb{C}^2/G,0)$.

Therefore, $(V(g),0)$ is isomorphic to $(V(f),0)$. In particular, $(V(g),0)$ is bi-Lipschitz homeomorphic to $(V(f),0)$. By previous results (Lemma 1), $\mathrm{rk}(V(f),0) = \mathrm{rk}(V(g),0) = 2$.
If $a = 0$, then $V(f)$ is the union of two planes P_1 and P_2. By A'Campo-Lê's Theorem, $V(g)$ is the union of two smooth surfaces, which also implies $\mathrm{rk}(V(f),0) = \mathrm{rk}(V(g),0) = 2$. By symmetry, if $\mathrm{rk}(V(f),0) = 1$, then $\mathrm{rk}(V(g),0) = 1$. Therefore, in any case, $\mathrm{rk}(V(g),0) = \mathrm{rk}(V(f),0)$, and thus $\mathrm{crk}(f) = \mathrm{crk}(g)$.

To finish our proof, we need the following lemma, which we state in a much more general setting.

Lemma *Let* $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be two holomorphic functions. Assume that* $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *is a generalized semi-Brieskorn-Pham function and there is a homeomorphism* $\phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *such that* $\phi(V(f)) = V(g)$. *If* $m(V(f), 0) > 2$ *or* $m_1(f)$ *is an odd number, then* $m(V(g), 0) \leq m(V(f), 0)$.

Proof

After a change of coordinates, if necessary, we may assume that f is written in the following way:

$$f(z_1, ..., z_r, z_{r+1}, ..., z_n) = z_1^{a_1} + ... + z_k^{a_k} + h(z_1, ..., z_n),$$

where $h\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a holomorphic function such that $\mathrm{ord}_0 h > a_k$. In this case, $in(f)(z) = z_1^a + ... + z_r^a$, where $r = m_1(f)$.

Let $q: : \mathbb{C}^r \to \mathbb{C}$ be the polynomial $q(z_1, ..., z_r) = z_1^a + ... + z_r^a$. Then $F_{in(f)} = F_q \times \mathbb{C}^{n-r}$. In particular, $\chi(F_{in(f)}) = \chi(F_q)$. However, $\chi(F_q) = 1 + (-1)^{r-1}(a-1)^r = 0$ if and only if $a = 2$ and r is an even number. Since we are assuming that $a > 2$ or r is an odd number, we have $\chi(F_{in(f)}) \neq 0$. Since the restriction to $F_{in(f)}$ of the natural projection $p: \mathbb{C}^n \setminus \{0\} \to \mathbb{C}P^{n-1}$ gives a finite covering map $p|: F_{in(f)} \to \mathbb{CP}^{n-1} \setminus PV(in(f))$, we obtain $\chi(\mathbb{CP}^{n-1} \setminus PV(in(f))) \neq 0$.

By the A'Campo Theorem, we obtain that

$$m(V(f),0) = \inf\{k > 0; \Lambda(h_f^k) \neq 0\}$$

and

$$m(V(g),0) \leq \inf\{k > 0; \Lambda(h_g^k) \neq 0\},$$

where $h_f : F_f \to F_f$ (resp. $h_g : F_g \to F_g$) is the monodromy of the Milnor fibration of f (resp. g), and $\Lambda(h)$ denotes the Lefschetz number of h.

Since $\Lambda(h_f^k) = \Lambda(h_g^k)$ for all k, we have that $m(V(g), 0) \leq m(V(f), 0)$.

Now, we are going to prove that $\mathrm{rk}(V(f),0) \neq 0$ if and only if $\mathrm{rk}(V(g),0) \neq 0$, which will complete the proof of the theorem. In fact this follows from a result of Aznar, but as far as we know, a proof of that result was never published. For convenience, we present a proof here.

Assume, for instance, that $\mathrm{rk}(V(f),0) \neq 0$. Then $m(V(f),0) = 2$, and thus f is a semi-Brieskorn-Pham function. In this case, $\mathrm{rk}(V(f),0) = m_1(f)$. If $\mathrm{rk}(V(f),0)$ is an odd number, then by the previous Lemma, $m(V(g),0) \leq m(V(f),0) = 2$. By A'Campo-Lê's Theorem, $m(V(g),0) \geq 2$. Therefore,
$m(V(g),0) = m(V(f),0) = 2$.
Thus, we may assume that $\mathrm{rk}(V(f),0) = 2$. By the Splitting Lemma again, we may assume that $f(x,y,z) = x^2 + y^2 + az^{k+1}$, where $k \geq 2$ and $a \in \{0,1\}$.

If $a = 1$, then by proceeding as in the proof of Claim 2, we obtain that $(V(g), 0)$ is isomorphic to $(V(f), 0)$. In particular, we obtain $m(V(g), 0) = m(V(f), 0) = 2$, and thus $\mathrm{rk}(V(g), 0) \neq 0$.
If $a = 0$, then $V(f)$ is the union of two planes P_1 and P_2. By A'Campo-Lê's Theorem, $V(g)$ is the union of two smooth surfaces. Then $m(V(g), 0) = 2$, and thus $\mathrm{rk}(V(g), 0) \neq 0$.

THANK YOU FOR THE ATTENTION!!!!!