Lipschitz geometry of non-isolated complex surface singularities

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• We say that $(X,0) \subset (\mathbb{C}^k,0)$ is **inner (resp. outer) bilipschitz equivalent** to $(Y,0) \subset (\mathbb{C}^r,0)$ if there exists a homeomorphism $h : (X,0) \longrightarrow (Y,0)$ which is bilipschitz with respect to the inner (resp. outer) metric.

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• We often write $(X,0) \sim_{d_i} (Y,0)$ (resp. $(X,0) \sim_{d_o} (Y,0)$) to say that $(X,0)$ is inner (resp. outer) bilipschitz equivalent to $(Y,0)$.

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$$(X_t, 0) = (\{xy(x-y)(x-ty) = 0\}, 0) \subset (\mathbb{C}^2, 0), |t| < 1.$$

Then

$$(X_{t_1}, 0) \sim_{Ann} (X_{t_2}, 0) \iff t_1 = t_2$$

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• C. 2026: Complete invariant for the inner Lipschitz classification of irreducible complex surface germs, possibly with non-isolated singularities.

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Let $(X, 0)$ be an irreducible complex surface germ. We say that $(X, 0)$ is **non pinched** if its normalization

$$n : (\overline{X}, 0) \longrightarrow (X, 0)$$

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• The Whitney umbrella, defined by $(X_{\mathrm{WU}}, 0) := (\{x^2 - y^2 z = 0\}, 0)$. is non pinched

$$\begin{array}{ccc} n\colon (\mathbb{C}^2, 0) & \longrightarrow & (X_{\mathrm{WU}}, 0) \subset (\mathbb{C}^3, 0), \\ (u, v) & \longmapsto & (uv,\ u,\ v^2). \end{array}$$

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- The cusp, defined by $(X_{\mathrm{cusp}}, 0) := (\{y^2 - x^3 = 0\}, 0)$. is pinched

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• There exists holomorphic functions $g_1, \ldots, g_r \in I(f_1, \ldots, f_k)$ on $(\overline{X},0)$ and $\lambda_{ij} \in \mathcal{O}_{\overline{X},0}$ such that

$$dg_i = \sum_{j=1}^{k} \lambda_{ij}\, df_j, \qquad i = 1, \ldots, n.$$

and the image of the map

$$\begin{array}{ccc} G : (\overline{X},0) & \longrightarrow & (X_G,0) \subset (\mathbb{C}^{k+r},0), \\ p & \longmapsto & (f_1(p), f_2(p), \ldots, f_k(p), g_1(p), \ldots, g_r(p)). \end{array}$$

is a complex surface germ with an isolated singularity and G is its normalization.

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We have the following commutative diagram:

$$\begin{array}{ccc} ((\overline{X},0),g_G) & \xrightarrow{G=(f_1,\ldots,f_k,g_1,\ldots,g_r)} & (X_G,0)\subset(\mathbb{C}^{k+r},0) \\ \Big\downarrow \mathrm{Id} & & \Big\downarrow P_k \\ ((\overline{X},0),g) & \xrightarrow{n=(f_1,\ldots,f_k)} & (X,0)\subset(\mathbb{C}^{k},0) \end{array}$$

where $(X_G,0)$ is a complex surface germ with an isolated singularity at the origin of $(\mathbb{C}^{k+r},0)$ and G is its normalization.

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- There exists a constant K such that: $K\left\|u\right\|_{g_G} \leq \left\|u\right\|_{g} \leq \left\|u\right\|_{g_G}$.

Thank you for your attention!

Dziękuję za uwagę!