Asymptotic lines and curvature lines defined by the Gauss map of surfaces in $\mathbb{R}^4$

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The Gauss map

Let M be a smooth oriented surface and let

$$F : M \longrightarrow \mathbb{R}^4$$

be an immersion.

The Gauss map assigns to each point its oriented tangent plane:

$$g : M \longrightarrow \mathrm{Gr}^+(2,4), \qquad g(p) = T_pM \subset \mathbb{R}^4.$$

Here $\mathrm{Gr}^+(2,4)$ denotes the Grassmannian of oriented two-dimensional subspaces of $\mathbb{R}^4$.

Plücker coordinates on $\mathrm{Gr}^+(2,4)$

Let (e_1, e_2, e_3, e_4) be a positive orthonormal basis of $\mathbb{R}^4$. If $V \subset \mathbb{R}^4$ is an oriented two-plane with positive orthonormal basis (v_1, v_2), write

$$\begin{aligned} v_1 \wedge v_2 = \alpha_{12} e_1 \wedge e_2 + \alpha_{23} e_2 \wedge e_3 + \alpha_{31} e_3 \wedge e_1 \\ + \alpha_{34} e_3 \wedge e_4 + \alpha_{14} e_1 \wedge e_4 + \alpha_{24} e_2 \wedge e_4. \end{aligned}$$

The image of the Grassmannian is characterized by

$$\alpha_{12}\alpha_{34} + \alpha_{31}\alpha_{14} + \alpha_{23}\alpha_{24} = 0, \qquad \sum \alpha_{ij}^2 = 1.$$

The product structure $\mathrm{Gr}^+(2,4) \simeq S^2 \times S^2$

An orthonormal basis of $\Lambda^2\mathbb{R}^4$ adapted to the Hodge splitting is

$$\begin{aligned} x_1 &= \tfrac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4), \quad y_1 = \tfrac{1}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4), \\ x_2 &= \tfrac{1}{\sqrt{2}}(e_2 \wedge e_3 + e_1 \wedge e_4), \quad y_2 = \tfrac{1}{\sqrt{2}}(e_2 \wedge e_3 - e_1 \wedge e_4), \\ x_3 &= \tfrac{1}{\sqrt{2}}(e_3 \wedge e_1 + e_2 \wedge e_4), \quad y_3 = \tfrac{1}{\sqrt{2}}(e_3 \wedge e_1 - e_2 \wedge e_4). \end{aligned}$$

If $v_1 \wedge v_2 = \sum_{i=1}^{3} \beta_i x_i + \gamma_i y_i$, then

$$\mathrm{Gr}^+(2,4) = \left\{ \sum \beta_i^2 = \frac{1}{2}, \quad \sum \gamma_i^2 = \frac{1}{2} \right\} = S_1 \times S_2.$$

Components of the Gauss map

Using the decomposition $\mathrm{Gr}^+(2,4) = S_1 \times S_2$, let

$$\pi_i : S_1 \times S_2 \longrightarrow S_i, \qquad i = 1, 2.$$

The two components of the Gauss map are

$$g_i = \pi_i \circ g : M \longrightarrow S_i, \qquad g = (g_1, g_2) : M \longrightarrow S_1 \times S_2.$$

These two spherical maps encode the tangent planes through the self-dual and anti-self-dual parts of $\Lambda^2\mathbb{R}^4$.

Darboux frames and the Gauss map

Let (e_1, e_2, e_3, e_4) be a local Darboux frame on $U \subset M$: it is positive orthonormal, and $(e_1(p), e_2(p))$ is a positive basis of T_pM. Then

$$g|_U = e_1 \wedge e_2.$$

Consequently,

$$g_1|_U = \frac{1}{2}(e_1 \wedge e_2 + e_3 \wedge e_4) = \frac{1}{\sqrt{2}}x_1,$$

$$g_2|_U = \frac{1}{2}(e_1 \wedge e_2 - e_3 \wedge e_4) = \frac{1}{\sqrt{2}}y_1.$$

Structure equations

For the immersion $F : M \to \mathbb{R}^4$, write

$$\mathrm{d}F = \theta_1 e_1 + \theta_2 e_2, \qquad \theta_i = \langle \mathrm{d}F, e_i \rangle, \quad i = 1, \ldots, 4.$$

The connection forms are defined by

$$\mathrm{d}e_i = \sum_{j=1}^{4} \omega_i^j e_j, \qquad \omega_i^j = \langle \mathrm{d}e_i, e_j \rangle.$$

They satisfy

$$\theta_3 = \theta_4 = 0, \qquad \omega_i^j = -\omega_j^i, \qquad \omega_i^i = 0.$$

The Gaussian curvature K and the normal curvature K_N are determined by

$$\mathrm{d}\omega_1^2 = -K\,\theta_1 \wedge \theta_2, \qquad \mathrm{d}\omega_3^4 = -K_N\,\theta_1 \wedge \theta_2.$$

Singular sets of the components

For the local expressions $g_1 = \frac{1}{\sqrt{2}} x_1$ and $g_2 = \frac{1}{\sqrt{2}} y_1$, one obtains

$$\mathrm{d}g_1 = \frac{1}{\sqrt{2}}(\omega_2^4 - \omega_1^3)x_2 + \frac{1}{\sqrt{2}}(-\omega_1^4 - \omega_2^3)x_3,$$
$$\mathrm{d}g_2 = \frac{1}{\sqrt{2}}(-\omega_2^4 - \omega_1^3)y_2 + \frac{1}{\sqrt{2}}(\omega_1^4 - \omega_2^3)y_3.$$

Moreover,

$$(K + K_N)\theta_1 \wedge \theta_2 = (\omega_1^3 - \omega_2^4) \wedge (\omega_1^4 + \omega_2^3),$$
$$(K - K_N)\theta_1 \wedge \theta_2 = (\omega_1^4 - \omega_2^3) \wedge (\omega_2^4 + \omega_1^3).$$

Jacobian of g_i

Definition

The Jacobian of a map $g_i : M \to S_i$ is a function $J(g_i) : M \to \mathbb{R}$ such that

$$g_i^* \sigma = J(g_i)\, \mathrm{d}A,$$

where σ is the area form on S_i and $\mathrm{d}A$ is the area form of M.

Theorem (Blaschke; Weiner; Hoffman–Osserman)

The Jacobians of the components of the Gauss map are

$$J(g_1) = \frac{1}{2}(K + K_N), \qquad J(g_2) = \frac{1}{2}(K - K_N).$$

Thus the singular set of g_i is

$$\Sigma_i = \{p \in M : K(p) - (-1)^i K_N(p) = 0\}.$$

Generic properties of g_i

For a generic immersion $M \to \mathbb{R}^4$, the following hold.

(1)
$$\mathrm{d}(K - (-1)^i K_N)\big|_{\{K-(-1)^i K_N=0\}} \neq 0,$$

so the singular set is smooth.

(2) The kernel line field
$$\{\omega_1^3 + (-1)^i \omega_2^4 = 0, \quad \omega_1^4 - (-1)^i \omega_2^3 = 0\}$$

is transverse to the singular curve, except at isolated points where it has first-order tangency.

These exceptional points are cusps; the remaining singular points are folds.

Generic self-intersections and stability

For generic g_i the following additional condition holds.

(3) If $p, q \in M$ are distinct points such that

$$K(p) - (-1)^i K_N(p) = K(q) - (-1)^i K_N(q) = 0, \qquad g_i(p) = g_i(q),$$

then the images of the tangent lines to the singular curve at p and q meet transversely in $T_{g_i(p)}S_i$. There are no triple points of this type.

Theorem (2024)

Properties (1)–(3) above are generic. If an immersion $M \to \mathbb{R}^4$ satisfies them, then g_i is stable.

A rigidity theorem

Theorem (Hoffman–Osserman)

Let M be connected. The following are equivalent:

$$g_i(p) = \text{constant for all } p \in M,$$

and there exists an orthogonal complex structure J on $\mathbb{R}^4$, preserving the orientation for $i = 1$ and reversing it for $i = 2$, such that M is a holomorphic curve in $\mathbb{C}^2$.

This identifies constant components of the Gauss map with complex curves for suitable orthogonal complex structures.

g_i-elliptic, hyperbolic and parabolic points

A point $p \in M$ is called

- g_i-elliptic if
$$K(p) + (-1)^{i+1} K_N(p) > 0;$$
- g_i-hyperbolic if
$$K(p) + (-1)^{i+1} K_N(p) < 0;$$
- g_i-parabolic if
$$K(p) + (-1)^{i+1} K_N(p) = 0.$$

Here $i = 1, 2$.

Contact with holomorphic curves

Locally, write M as the graph of a map

$$(a, b) : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0),$$

with first derivatives zero at the origin. Compare M with the graph of a holomorphic function

$$\varphi : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0).$$

The contact map germ is

$$k(x, y) = (a(x, y) - \operatorname{Re}\varphi(x + iy), b(x, y) - \operatorname{Im}\varphi(x + iy)).$$

Two contact germs $k, \widetilde{k}$ are $\mathcal{K}$-equivalent if

$$\widetilde{k}(x, y) = A(x, y)k(\Phi(x, y)),$$

where Φ is a diffeomorphism germ and $A(x, y)$ is an invertible matrix germ.

Singularity type via contact

Theorem (2024)

Let k be a contact map germ of M and a holomorphic curve with the highest contact in s-jets at p.

1. *p is g_1-elliptic iff k is contact equivalent to a germ with two-jet $(x^2 + y^2, 0)$.*
2. *p is g_1-hyperbolic iff $j^2k \sim (xy, 0)$.*
3. *p is a corank-one singular point of g_1 iff $j^2k \sim (x^2, 0)$.*
4. *p is a fold of g_1 iff $j^3k \sim (x^2 + y^3, 0)$.*
5. *p is a cusp of g_1 iff $j^4k \sim (x^2 \pm y^4, 0)$.*
6. *p is a corank-two singular point of g_1 iff $j^2k \sim (0, 0)$.*

Mean curvature and a bilinear form

The mean curvature vector is

$$H = \frac{1}{2}(\omega_1^3(e_1) + \omega_2^3(e_2))e_3 + \frac{1}{2}(\omega_1^4(e_1) + \omega_2^4(e_2))e_4.$$

Let $H^\perp$ be a normal vector field such that

$$\langle H^\perp, H\rangle = 0, \qquad |H^\perp| = |H|,$$

and, where $H \neq 0$, the pair $(H, H^\perp)$ is a positive basis of N_pM.
For $v, w \in T_pM$ define

$$B_{g_i}(v, w) = \langle \mathrm{d}g_i(v), H^\perp \wedge w\rangle, \qquad i = 1, 2.$$

Then B_{g_i} is a smooth symmetric bilinear form.

Quadratic forms associated with g_i

Definition

The quadratic form of g_i is

$$Q_{g_i}(v) = \langle \mathrm{d}g_i(v), H^{\perp} \wedge v \rangle = a\theta_1^2 + 2b\theta_1\theta_2 + c\theta_2^2.$$

Proposition

Its discriminant is

$$\Delta(Q_{g_i}) = -\frac{1}{4}|H|^2(K + (-1)^{i+1}K_N), \qquad i = 1, 2.$$

Thus the sign of $K + (-1)^{i+1}K_N$ controls the type of Q_{g_i}.

Relation with the first fundamental form

Let I_M be the first fundamental form on M. Then

$$Q_{g_1} = \frac{1}{2}\left(g_1^* I_{S_1} + \frac{1}{2}(K + K_N) I_M\right),$$

$$Q_{g_2} = -\frac{1}{2}\left(g_2^* I_{S_2} + \frac{1}{2}(K - K_N) I_M\right).$$

These formulas relate the quadratic forms defining g_i-asymptotic lines to the pullbacks of the standard metrics on the two spheres.

Classical relation and total quadratic form

Let II_M be the second fundamental form and let

$$III_M(v) = \langle \mathrm{d}g(v), \mathrm{d}g(v) \rangle$$

be the third fundamental form. The classical relation is

$$III_M + K\, I_M = 2\langle H, II_M \rangle.$$

Consequently,

$$\langle H, II_M \rangle = Q_{g_1} - Q_{g_2},$$

$$g_1^* I_{S_1} - g_2^* I_{S_2} + K_N I_M = 2Q_g, \qquad Q_g(v) = \langle \mathrm{d}g(v), H^\perp \wedge v \rangle = Q_{g_1} + Q_{g_2}.$$

A holomorphic-curve interpretation

Theorem

Let $p \in M$.

1. *If $H_p \neq 0$, then there exists a J-holomorphic curve C_i, for a complex orthogonal structure J on $\mathbb{R}^4$, preserving orientation for $i = 1$ and reversing orientation for $i = 2$, such that*

$$Q_{g_i}|_p \otimes \frac{2H_p}{|H_p|^3} = (-1)^i (II_M - II_{C_i})|_p.$$

   Moreover $II_{C_i}|_p$ is determined by

$$\langle H^\perp, II_{C_i} \rangle|_p = \langle H^\perp, II_M \rangle|_p.$$

2. *If $H_p = 0$, then for every $0 \neq v \in N_pM$ there exists such a holomorphic curve C with*

$$\langle v, II_C \rangle|_p = \langle v, II_M \rangle|_p.$$

g_i-asymptotic directions

Assume $p \in M$, $H_p \neq 0$, and $0 \neq v \in T_pM$.

Definition

The line span(v) is a g_i-asymptotic direction at p if

$$Q_{g_i}(v) = 0.$$

The integral curves of these directions are called g_i-asymptotic lines.

If p is g_i-hyperbolic, then there are two g_i-asymptotic directions at p. If p is g_i-parabolic, then there is one g_i-asymptotic direction at p.

Local equations for g_1-asymptotic lines

Let $H_p \neq 0$ and let (e_1, e_2, e_3, e_4) be a positive Darboux frame with

$$e_3 = \frac{H}{|H|}.$$

Assume $K(p) + K_N(p) \leq 0$. Then

$$Q_{g_1} = \frac{1}{4}(\omega_1^3(e_1) + \omega_2^3(e_2))\left((\omega_1^3 - \omega_2^4)\theta_1 + (\omega_2^3 + \omega_1^4)\theta_2\right).$$

Hence, when the indicated coefficients are non-zero, the g_1-asymptotic lines are given by

$$\omega_1^3 - \omega_2^4 \pm \sqrt{-(K + K_N)}\,\theta_2 = 0,$$

or equivalently by

$$\omega_2^3 + \omega_1^4 \pm \sqrt{-(K + K_N)}\,\theta_1 = 0.$$

Folds, cusps and the quadratic form

Theorem

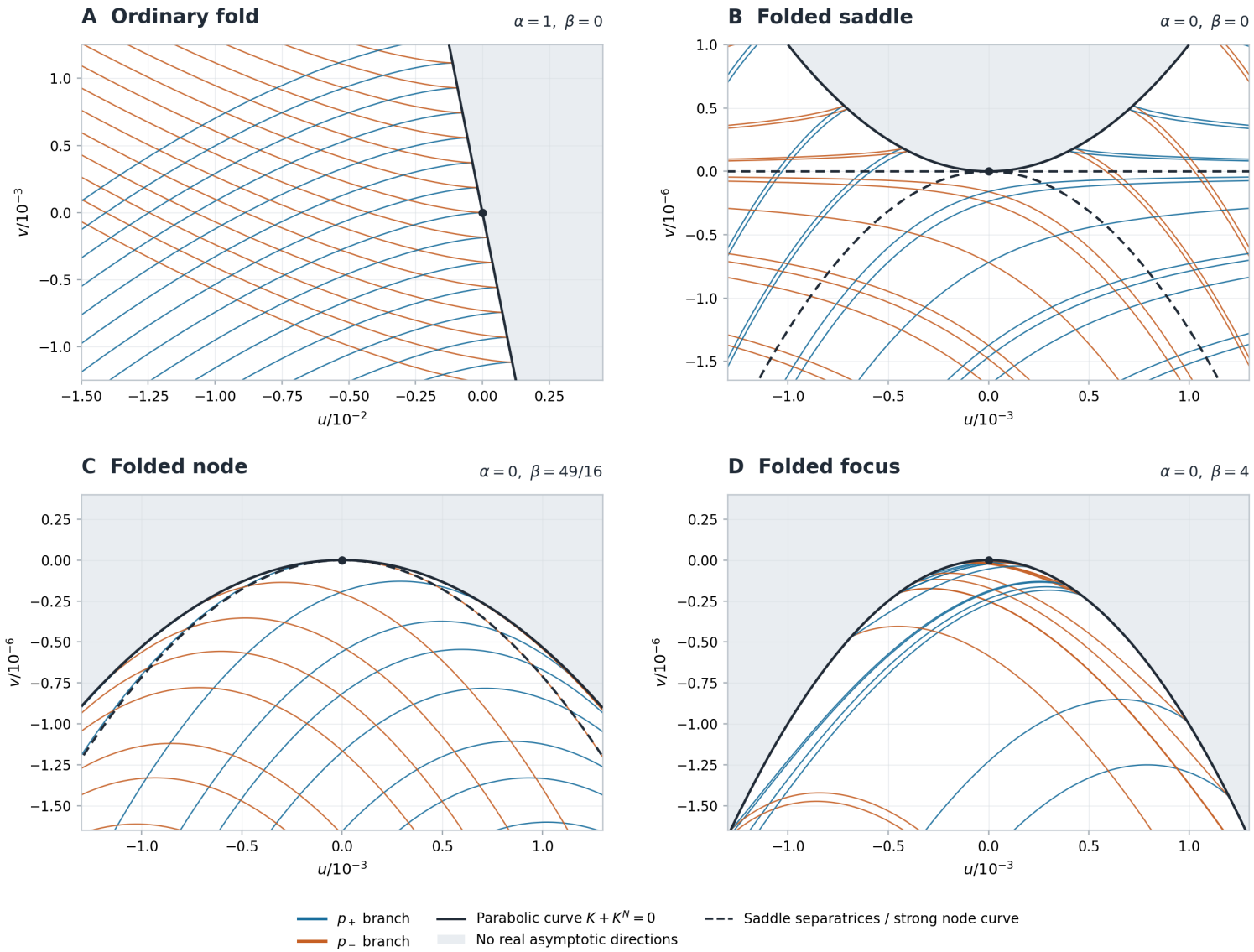
Let $p \in M$ and $H_p \neq 0$.

1. *p is a simple fold of Q_{g_i} if and only if p is a fold of g_i.*
2. *If p is a folded saddle of Q_{g_i}, then p is a negative cusp of g_i.*
3. *If p is a folded node or focus of Q_{g_i}, then p is a positive cusp of g_i.*

The local dynamics of asymptotic lines therefore detects the singular type of the Gauss map component.

g_1-asymptotic foliations

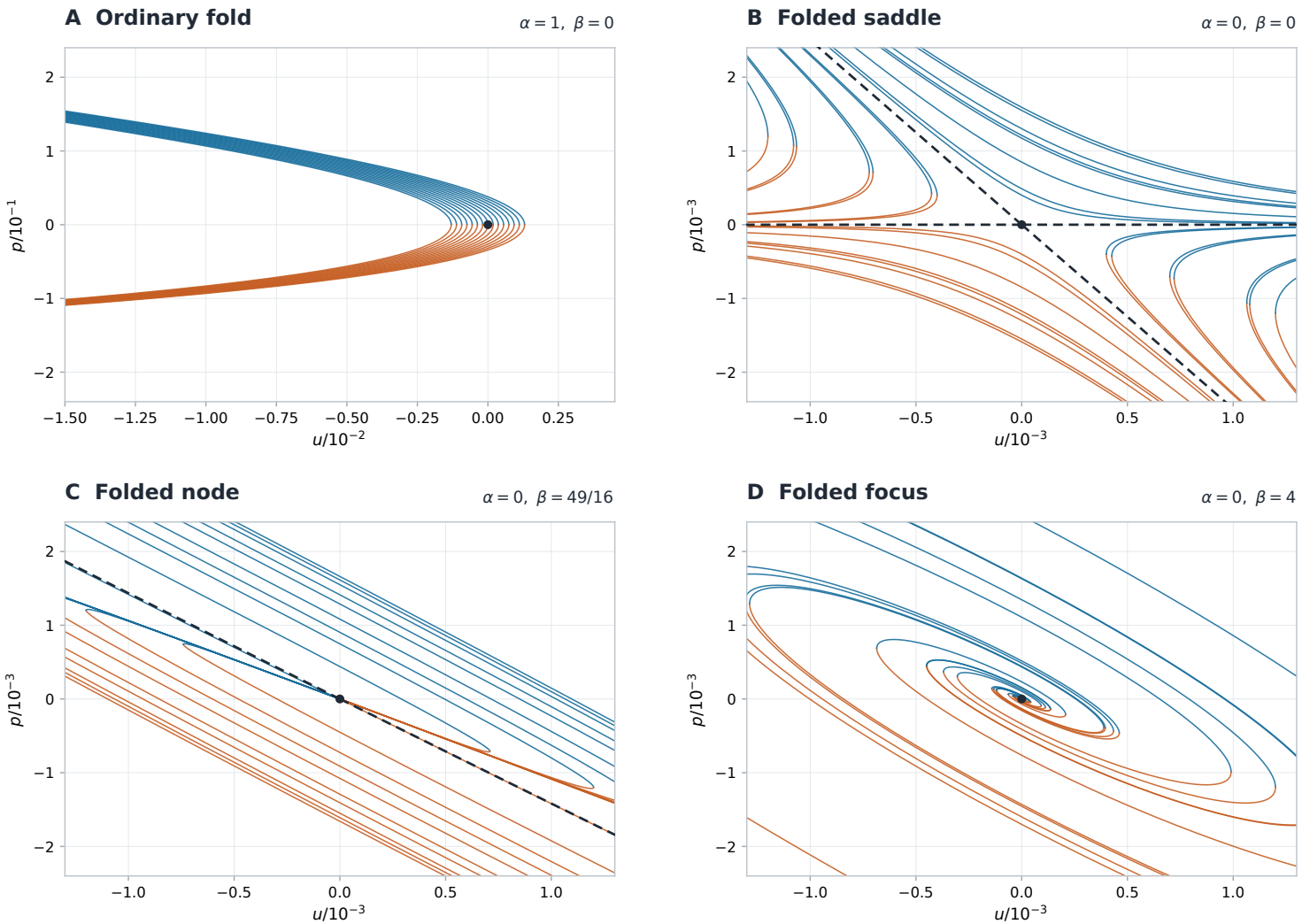
$\mathbf{x}^{\alpha,\beta}(u,v) = (u, v, \frac{u^2}{2} + \frac{uv(u+v)}{2} + \frac{\alpha u^3}{6} + \frac{\beta u^4}{24}, uv)$



Numerical integral curves of the exact BDE. Axes use different scales; these are local coordinate portraits.

Lifted phase portraits

Coordinates (u, p) on $F = a + 2bp + cp^2 = 0$, with $p = dv/du$; each trajectory projects to a leaf.



Numerical integral curves of the exact BDE. Axes use different scales; these are local coordinate portraits.

Curvature directions from the Gauss map

For a normal field ν, define the quadratic form

$$\mathcal{C}_\nu(v) = \langle dg(v), \nu \wedge v \rangle, v \in T_pM. \tag{1}$$

Let J_T be the positive tangent rotation, $J_T e_1 = e_2$. Then

$$\mathcal{C}_\nu(v) = -\langle II(v, J_T v), \nu \rangle. \tag{2}$$

A nonzero vector v satisfies $\mathcal{C}_\nu(v) = 0$ if and only if it is an eigenvector of the shape operator S_ν, defined by $I(S_\nu v, w) = \langle II(v, w), \nu \rangle$.
Taking $\nu = H^\perp$ gives $\mathcal{C}_{H^\perp} = Q_{g_1} + Q_{g_2}$.

Definition

For $H(p) \neq 0$ and $S_{H^\perp} \neq 0$, the g-curvature lines are the integral curves of the two $H^\perp$-principal direction fields.

Gauss–Bonnet type formula I

Theorem (2024)

Let M be an oriented closed surface, and let $M \to \mathbb{R}^4$ be an immersion satisfying the generic properties above. Then

$$2\pi\chi(M) = \int_M |K \pm K_N| \, \mathrm{d}A + 2\int_{\{K \pm K_N = 0\}} \kappa_g(\alpha) \, \mathrm{d}\alpha.$$

Here κ_g denotes the geodesic curvature contribution along the singular curve of the corresponding component of the Gauss map.

Gauss–Bonnet type formula II

For the same class of immersions,

$$\frac{1}{\pi}\int_M K \,\mathrm{d}A = \sum_{i=1}^{2}\left(\chi(M_i^+) - \chi(M_i^-) + S_{g_i}^+ - S_{g_i}^-\right),$$

$$\frac{1}{\pi}\int_M K_N \,\mathrm{d}A = \sum_{i=1}^{2}(-1)^{i-1}\left(\chi(M_i^+) - \chi(M_i^-) + S_{g_i}^+ - S_{g_i}^-\right),$$

where

$$M_i^+ = \{K > (-1)^{i-1}K_N\}, \qquad M_i^- = \{K < (-1)^{i-1}K_N\}.$$

The quantities $S_{g_i}^+$ and $S_{g_i}^-$ count positive and negative cusp contributions.

Embedding corollary

If M is an oriented closed surface and $M \to \mathbb{R}^4$ is an embedding satisfying the generic properties, then for $i = 1, 2$,

$$\chi(M) = \chi(M_i^+) + \frac{1}{2}\left(S_{g_i}^+ - S_{g_i}^-\right),$$

$$\chi(M_i^-) = \frac{1}{2}\left(S_{g_i}^+ - S_{g_i}^-\right).$$

These identities express the topology of the elliptic and hyperbolic regions through cusp counts of the Gauss map components.

Reference

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