On topological invariance of the Łojasiewicz exponent

Szymon Brzostowski and Tadeusz Krasiński

University of Łódź

Łojasiewicz100

Uniwersytet Jagielloński

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The Zariski multiplicity conjecture

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Definition. f, g are topologically equivalent (= V-equivalent) if there exists a homeomorphism

$$(\boldsymbol{C}^n, V(f), 0) \to (\boldsymbol{C}^n, V(g), 0).$$

Notation

$$f \overset{\text{V-top.}}{\sim} g$$

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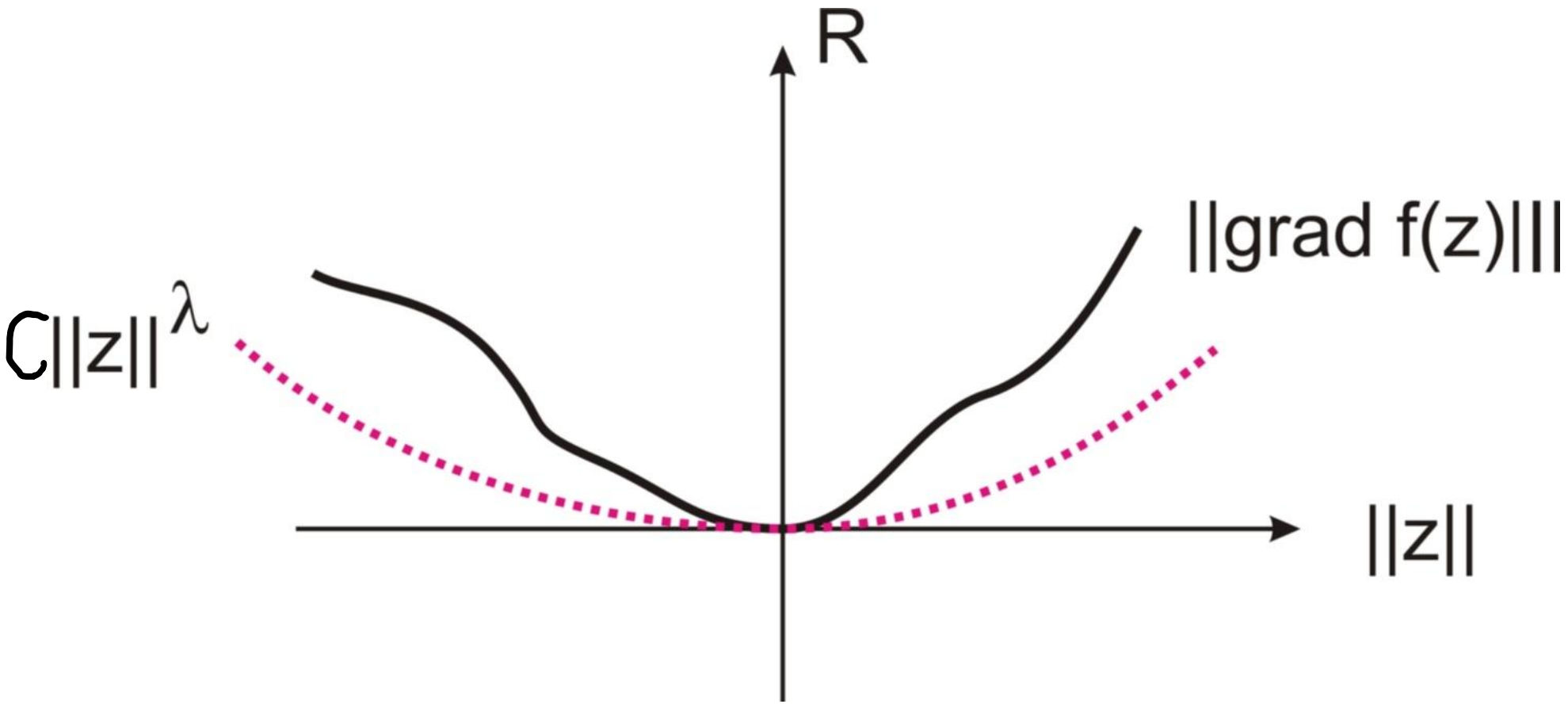
Definition. The Łojasiewicz exponent $\mathcal{L}(f)$ of an isolated singularity f is

$$\mathcal{L}(f) := \inf\{\lambda\colon \left|\left|\nabla(f(\boldsymbol{z})\right|\right| \geq C\ ||\boldsymbol{z}||^{\lambda}\}$$

in a neighbourhood of the origin in $\boldsymbol{C}^n$ for some $C > 0$, where

$\nabla f(z) := (\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z)) =: \mathrm{grad} f(z)$

The Łojasiewicz exponent of a singularity



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5. Many mathematicians were engaged in studies of $\mathcal{L}(f)$.

Information on the CŁ.

1. Theorem. CŁ is true for $n = 2$ (for plane curve singularities). It follows from a formula for $\mathcal{L}(f)$ of $f: (\boldsymbol{C}^2, 0) \to (\boldsymbol{C}, 0)$ in terms of characteristics of branches of f and their mutual intersections: if $f = f_1 \dots f_k$ then

$$\mathcal{L}(f) = F(\beta(f_1), \dots, \beta(f_k), (f_i, f_j)).$$

(Kuo, Lu 1977, Płoski 2001). Since $\beta(f_1), \dots, \beta(f_k), (f_i, f_j))$ are topological invariants of f then $\mathcal{L}(f)$ as well.

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2. Theorem. (C. Bivia-Ausina 2017) CŁ is true in bilipschitz category.

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4. Reduced CŁ (B. Teissier) If (f_s) is μ-constant family of isolated singularities then (f_s) is $\mathcal{L}$-constant family.

Negative solution of the CŁ.

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Theorem. $\mathcal{L}(f)$ is not a topological invariant of isolated singularities $f\colon (\boldsymbol{C}^n, 0) \to (\boldsymbol{C}, 0)$ for $n \geq 4$.

Corollary. CŁ is not true.

Seifert form

The Seifert form of an isolated singularity $f\colon (\boldsymbol{C}^n, 0) \to (\boldsymbol{C}, 0)$ is a $\mathbb{Z}$-bilinear form $L\colon H_{n-1}(F;\mathbb{Z}) \times H_{n-1}(F;\mathbb{Z}) \to \mathbb{Z}$, where F is the Milnor fiber of f.

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3. It is a general notion from higher-dimensional knot theory.
4. Formal definition is

$$L(a,b) := LinkingNumber_{S^{2n-1}}\left(a, \tilde{b}\right),$$

where $\tilde{b}$ is a monodromy shifting of the cycle b.

Results of Philippe du Bois and Françoise Michel

1. There are plane curve singularities f and g :$(\boldsymbol{C}^2, 0) \to (\boldsymbol{C}, 0)$ of different topological type i.e. $f \overset{\text{V-top.}}{\nsim} g$ (exactly they have different dual resolution graphs) such that $L_f = L_g$.

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2. They noticed a known result: Seifert form is a complete topological invariant for $n \geq 4$ i.e.

$$F \overset{\text{V-top.}}{\sim} G \;\Leftrightarrow\; L_F \overset{\text{over } \mathbb{Z}}{\cong} L_G.$$

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3. They noticed next known result: Suspension preserves the Seifert form i.e. if $L_{f(x,y)} = L_{g(x,y)}$ and $h(z_1, \dots, z_n)$: $(\boldsymbol{C}^n, 0) \to (\boldsymbol{C}, 0)$ is an additional isolated singularity then $L_{f \oplus h} \cong L_{g \oplus h}$, where $f \oplus h, g \oplus h$ are direct sum of sing. (Sakamoto 1974).

Results of Philippe du Bois and Françoise Michel

4. In particular for $h(z,w) = z^2 + w^2$ if $L_f = L_g$ then we have $L_{f \oplus h} \underset{\text{over } \mathbb{Z}}{\cong} L_{g \oplus h}$.

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6. It is easy to see that the Łojasiewicz exponent is invariant of the above suspension

$$\mathcal{L}(f) = \mathcal{L}(f(x,y) + z^2 + w^2),$$
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7. Assuming (CŁ) we get from item 5 and 6

$$\mathcal{L}(f) = \mathcal{L}(g).$$

Results of Philippe du Bois and Françoise Michel

8. In the families of pairs (f, g) constructed by P. du Bois and F. Michel with different topological types and the same Seifert forms there are such ones which have different highest polar invariants. But the latter determine the Łojasiewicz exponents of f and g. So for such pairs

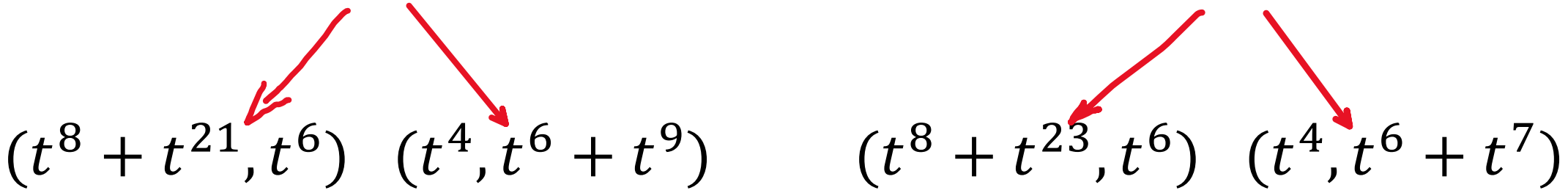
$$\mathcal{L}(f) \neq \mathcal{L}(g).$$

This contradicts equality in item 7.

Results of Philippe du Bois and Françoise Michel

9. An example of such a pair:

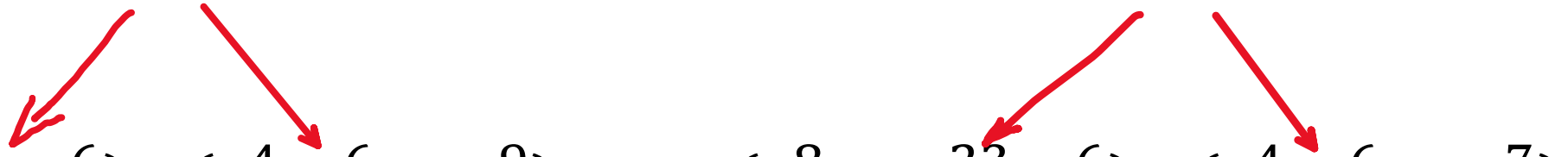
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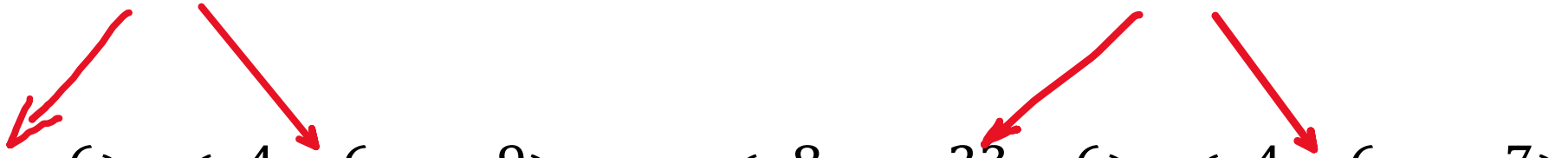
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Equations:

$f = ((x^3 - y^4)^2 - 3x^4y^7 - 6xy^{11} + 3x^2y^{14} - y^{21})((x^3 - y^2)^2 - 4x^6y - x^9)$

$g = ((x^3 - y^4)^2 - 6x^3y^9 - 3y^{13} + 3y^{18} - y^{23})((x^3 - y^2)^2 - x^7 - 4x^5y)$

Results of Philippe du Bois and Françoise Michel

For these functions f and g putting

$$F(x,y,z,w) \coloneqq f(x,y) + z^2 + w^2,$$
$$G(x,y,z,w) \coloneqq g(x,y) + z^2 + w^2$$

we have

$$F \overset{\text{V-top.}}{\sim} G$$

and

$$\mathcal{L}(F) = 16\frac{1}{3},$$
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Problems

Problem 1. In the plane $(n = 2)$ the Łojasiewicz exponent is topological invariant. For $n \geq 4$ the Łojasiewicz exponent is not topological invariant. Decide what is in the case $n = 3$.

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Problem 3. For the above examples of $f(x, y)$ and $g(x, y)$ for which:

1. $f(x, y)$ and $g(x, y)$ are not topologically equivalent,
2. $f(x, y) + z^2 + w^2$ and $g(x, y) + z^2 + w^2$ are topologically equivalent,

find a direct proof of 2-nd assertion.

Seifert form

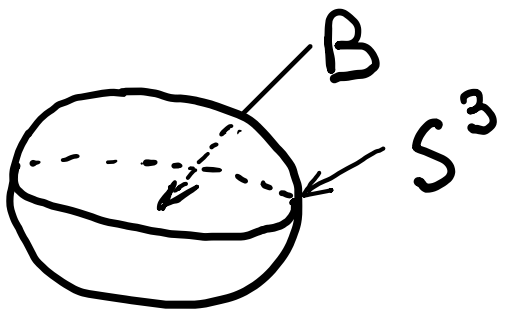
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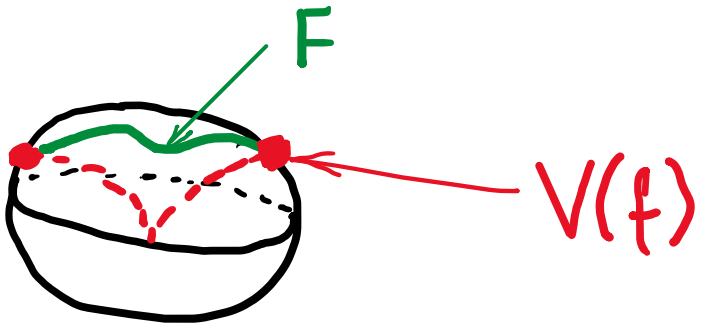
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B – small ball around 0, $\partial B = S^3$- three dimensional sphere

$\Phi: S^3 \setminus V(f) \to S^1$, $\Phi(x, y) := f(x, y)/|f(x, y)|$ - Milnor fibration

$F := \Phi^{-1}(1)$ – the Milnor fiber of f, $H_1(F; \mathbb{Z}) \cong \mathbb{Z}^{\mu}$





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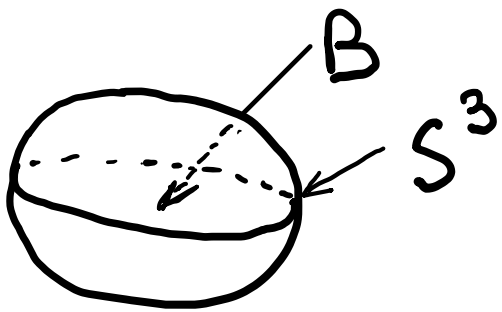
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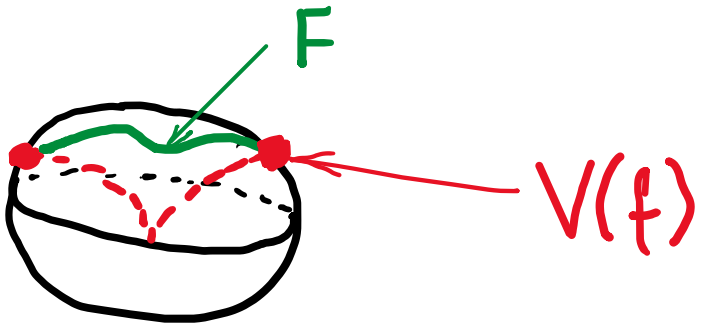
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$h_t: \Phi^{-1}(1) \to \Phi^{-1}(e^{2\pi i t})$ – the geometric monodromy for $t \epsilon [0,1]$

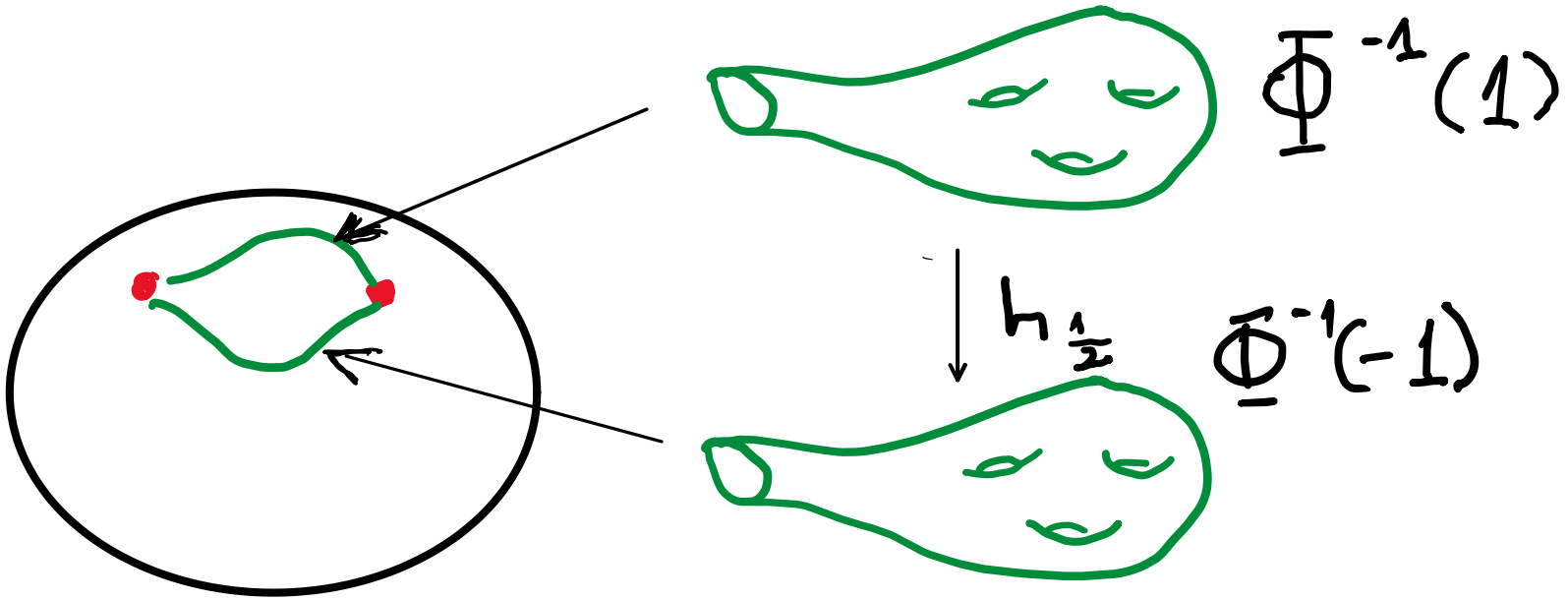
$h_{t*}: H_1(F; \mathbb{Z}) \to H_1(\Phi^{-1}(e^{2\pi i t}); \mathbb{Z})$ – the monodromy induced by h_t

Seifert form

The Seifert form

$$L: H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$$
$$L(a,b) = Lk_{S^3}(a, h_{\frac{1}{2}*}(b))$$

Lk_{S^3} - the linking number of cycles representig (a,b) in S^3



The end

Thank you for your attention.

