Combinatorial type and splitting invariants of plane curves

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Łojasiewicz 100 - Analytic Geometry and Singularity Theory

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§1 Introduction

motivation

§1 Introduction

In this talk, the base field is $\mathbb{C}$.

Let $\mathcal{C}$ be a plane curve $\mathcal{C} \subset \mathbb{P}^2$ (not necessarily irreducible).

We consider the embedded topology of $\mathcal{C}$ in $\mathbb{P}^2$,

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Theorem (Łojasiewicz, 1964)

There exists a finite triangulation of $\mathbb{P}^2$ such that each irreducible components of $\mathcal{C}$ is a subcomplex.

We can define a regular neighborhood $U(\mathcal{C})$ of $\mathcal{C}$ as the closed star of $\mathcal{C}$ in the second barycentric subdivision for a triangulation of $\mathbb{P}^2$ in the above theorem.

§1 Introduction

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- the number of irred. components,
- the degree and types of singularities of each irred. component,
- intersections of irred. components.

Fact

Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ be two plane curves.

(i) $(U(\mathcal{C}_1), \mathcal{C}_1) \cong (U(\mathcal{C}_2), \mathcal{C}_2)$ *: homeo. for reg. nbds.* $U(\mathcal{C}_i) \subset \mathbb{P}^2$ $\iff$ $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *have the same combinatorial type.*

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- Combinatorial type does NOT determine embedded topology (e.g. [Zariski, 1929], [Artal, 1994]).

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E : a smooth cubic,
$P_1, P_2, P_3, P_4 \in E$: inflectional points s.t. P_1, P_2, P_3 are collinear,
L_i : the tangent line of E at P_i $(i = 1, \ldots, 4)$,

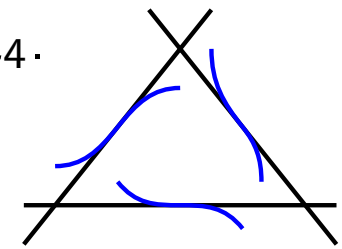
$$\mathcal{A}_1 := E + L_1 + L_2 + L_3, \quad \mathcal{A}_2 := E + L_1 + L_2 + L_4.$$

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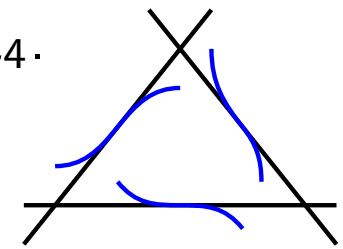
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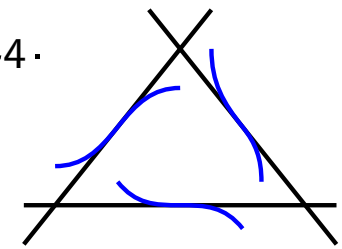
But $(\mathbb{P}^2, \mathcal{A}_1) \ncong (\mathbb{P}^2, \mathcal{A}_2)$ (by proving $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_1) \ncong \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2)$).

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Definition (cf. Artal, '94)

A tuple $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ of plane curves is a *Zariski k-tuple* if for $i \neq j$,

- $\mathcal{C}_i, \mathcal{C}_j$ have the same combinatorial type, and
- $\mathcal{C}_i, \mathcal{C}_j$ have different embedded topology,

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Problem

Classify the embedded topology of plane curves with a fixed combinatorial type.

The fundamental group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is a nice invariant.

But, $^{\exists}(\mathcal{C}_1, \mathcal{C}_2)$: a Zariski pair s.t. $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_1) \cong \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_2)$.

It is important to introduce a nice invariant.

§1 Introduction

Known invariants

Let $\mathcal{C} \subset \mathbb{P}^2$ be a plane curve.

- Fundamental group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$, and related invariants.
- Linking invariant [B. Guerville-Ballé and J.-B. Meilhan, '17].
- For line arrangements,
  loop-linking number [W. Cadegan-Schlieper, '18].
  (Its generalization "homology inclusion" is introduced in
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Aim:

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Aim: We generalize splitting invariants.

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Let $\sigma\colon Y \to \mathbb{P}^2$ be the minimal good embedded resolution of $\mathcal{C}$.

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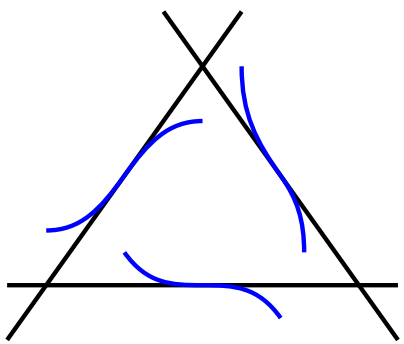
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e.g. (Artal's example, [Artal, '94])

If $\mathcal{C} = E + L_1 + L_2 + L_3$ with E a sm. cubic and L_i inflectional tangents of E, then $\mathrm{Cmb}(\mathcal{C})$ is constructed as follows:



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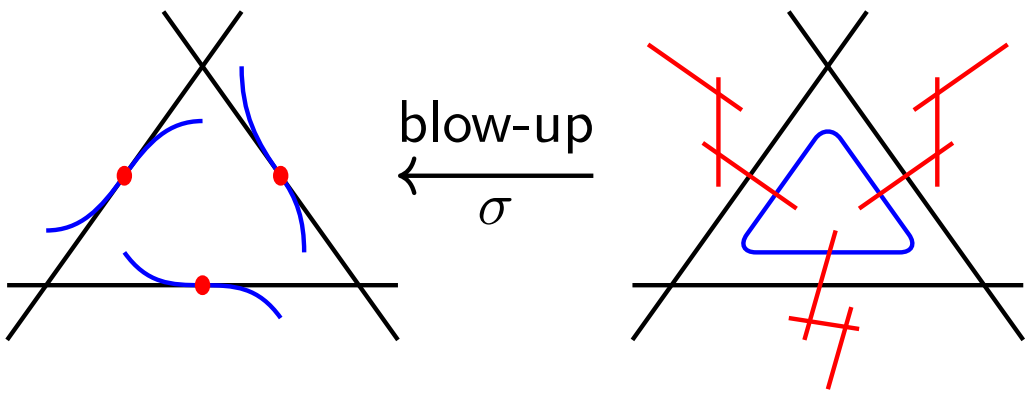
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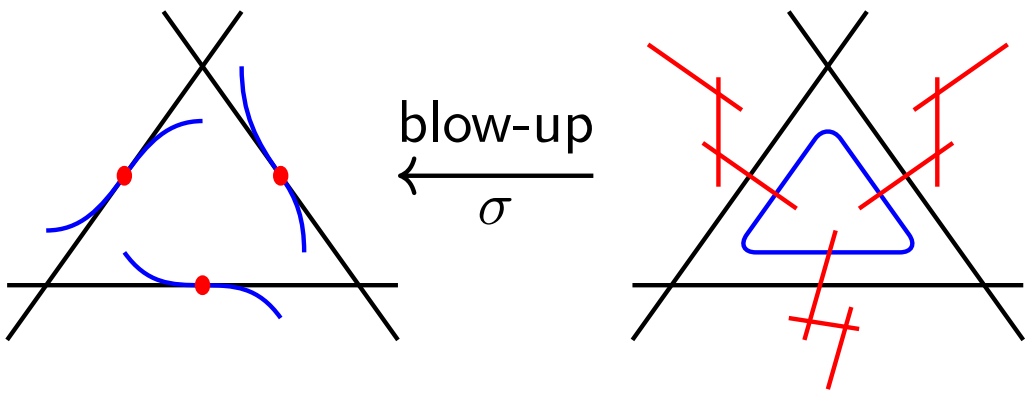
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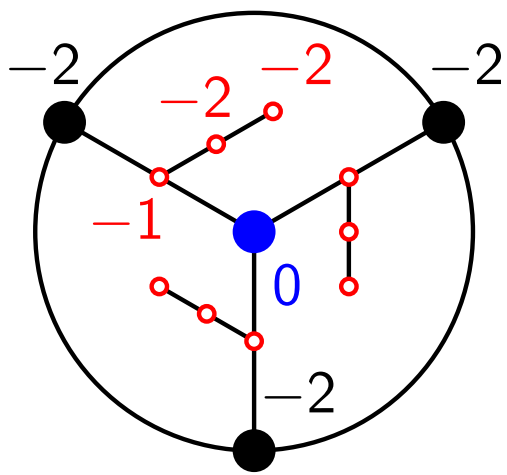
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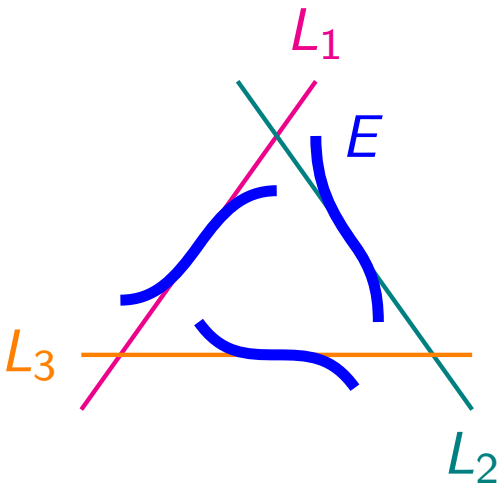
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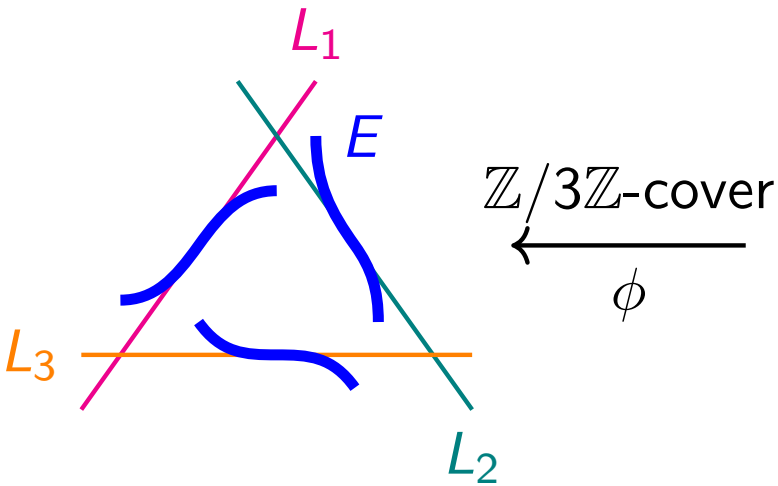
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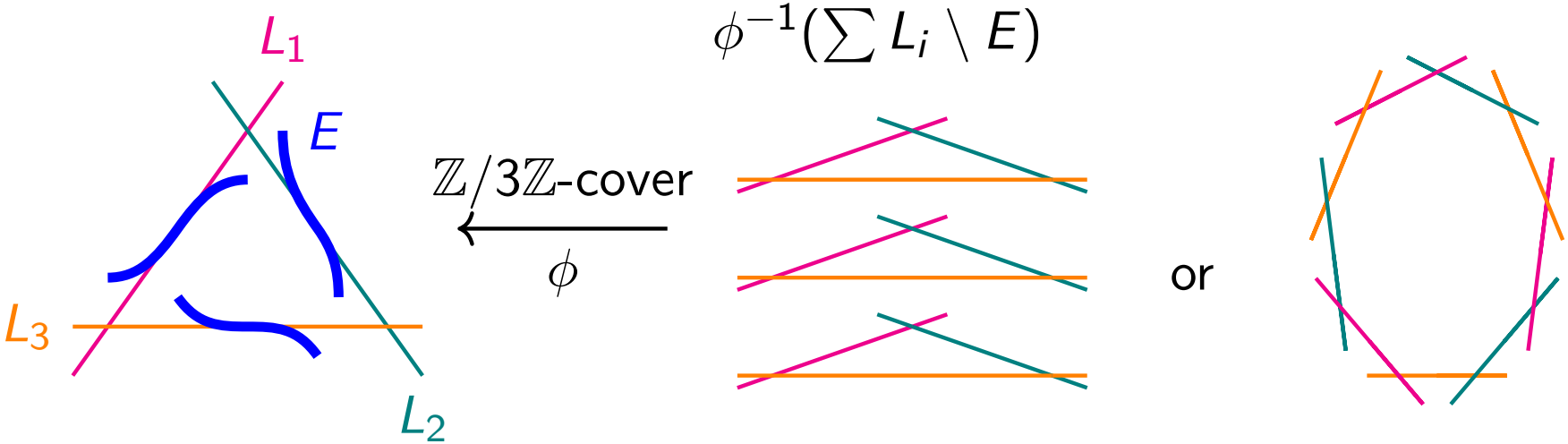
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Assume that $^{\exists}h \colon \mathbb{P}^2 \to \mathbb{P}^2$: homeo. and $^{\exists}\tau \colon G \to G$: auto.
s.t. $h(B_1) = B_2$, $h(D_1) = D_2$ and $\tau \circ \theta_1 = \theta_2 \circ h_*$.

$$\begin{array}{ccc} \pi_1(\mathbb{P}^2 \setminus B_1) & \xrightarrow{\theta_1} & G \\ \Big\downarrow{\scriptstyle h_*} & & \Big\downarrow{\scriptstyle \tau} \\ \pi_1(\mathbb{P}^2 \setminus B_2) & \xrightarrow[\theta_2]{} & G \end{array}$$

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Then we have the following diagram:

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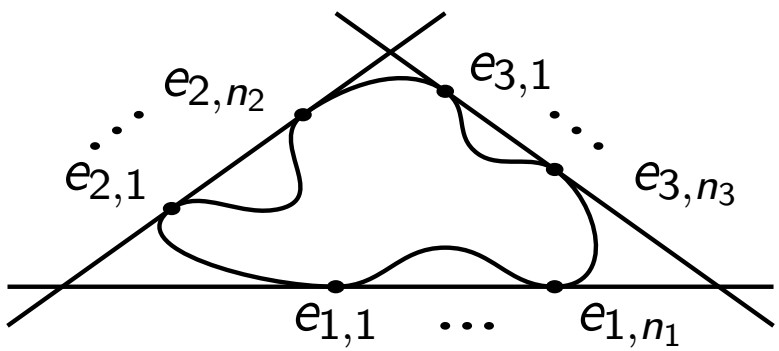
$\tilde{h}$ induces $\phi_1^{-1}(D_1 \setminus B_1) \xrightarrow{\sim} \phi_2^{-1}(D_2 \setminus B_2)$: G-eq. homeo. w.r.t. τ.

§3 Splitting invariants

Definition (AS curve of type $(d; \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$)

$B \subset \mathbb{P}^2$: a smooth curve of degree $d \geq 3$

$\mathfrak{p}_i = (e_{i,1}, \ldots, e_{i,n_i})$: three partitions of d $(i = 1, 2, 3)$



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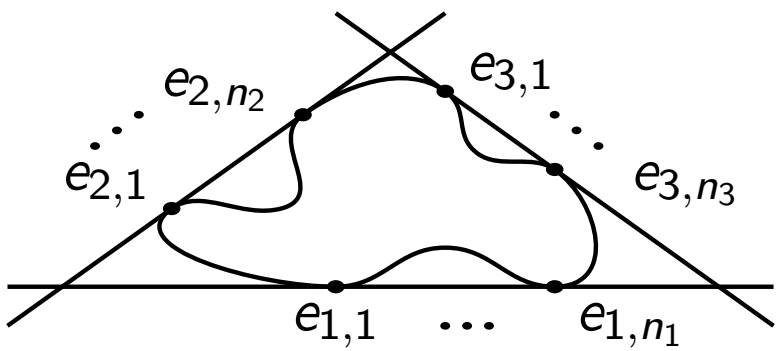
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If $^{\exists} L_i \subset \mathbb{P}^2$ $(i = 1, 2, 3)$: three lines s.t.

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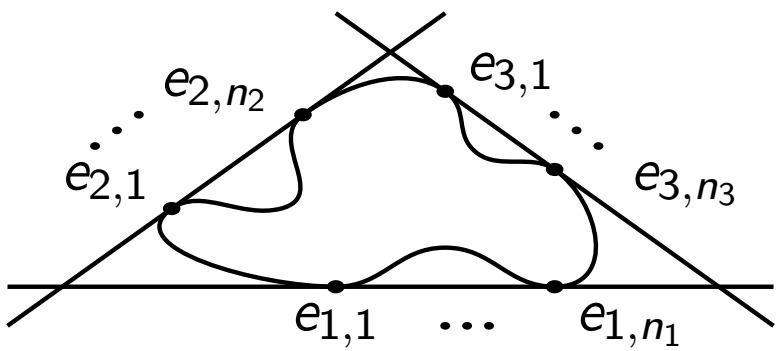
§3 Splitting invariants

Definition (AS curve of type $(d; \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$)

$B \subset \mathbb{P}^2$: a smooth curve of degree $d \geq 3$
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then $\mathcal{A} := B + L_1 + L_2 + L_3$ is called
an Artal–Shirane (AS) curve of type $(d; \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$.

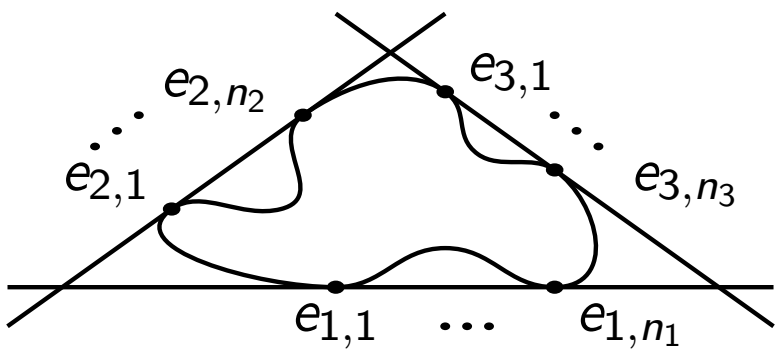


§3 Splitting invariants

Note that AS curves of type $(3; (3), (3), (3))$ are Artal's example.

Theorem (S., 2019)

The embedded topology of AS curves is classified by splitting invariants.



§3 Splitting invariants

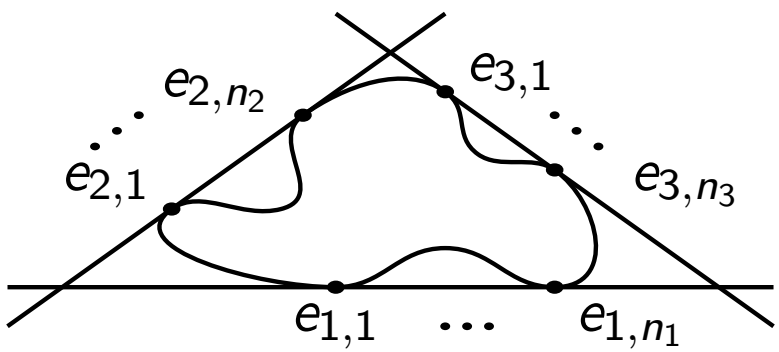
Note that AS curves of type $(3;(3),(3),(3))$ are Artal's example.

Theorem (S., 2019)

The embedded topology of AS curves is classified by splitting invariants.

Theorem (Artal–Cogolludo–Martín, 2020)

If $\mathcal{C} \subset \mathbb{P}^2$ is an AS curve of type $(d;\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)$ with $d > 3$, then $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is abelian, hence, $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \cong \mathbb{Z}^3$.



§3 Splitting invariants

Aim: We extend splitting invariants to an invariant containing data of ramification loci.

Problem:

Idea: (1)

(2)

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Idea: (1) We consider the "boundary manifold" of $\mathcal{C}_i \subset \mathbb{P}^2$, and study its image under a homeo. $h\colon (\mathbb{P}^2, \mathcal{C}_1) \to (\mathbb{P}^2, \mathcal{C}_2)$ by using arguments of graph manifolds [Waldhausen, '67] and plumbing graphs [Neumann, '81].
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(2) Describe the "combinatorial type" of $\phi_i^{-1}(\mathcal{C}_i)$ for a G-cover $\phi_i\colon X_i \to \mathbb{P}^2$ by modified plumbing graphs [Hironaka,'00].

§4 G-combinatorial type 1 (roughly)

A new invariant and examples

§4 G-combinatorial type 1

Let $\mathcal{C} \subset \mathbb{P}^2$ be a plane curve with $\theta\colon \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \twoheadrightarrow G$ (G: fin. gp.), let $\sigma\colon Y \to \mathbb{P}^2$ be the minimal good embedded resolution of $\mathcal{C}$, $\widetilde{\mathcal{C}} := \sigma^{-1}(\mathcal{C})$.

$$\mathcal{C} \subset \mathbb{P}^2 \xleftarrow[\text{emb. resol.}]{\sigma} Y \supset \widetilde{\mathcal{C}} := \sigma^{-1}(\mathcal{C})$$

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$$\begin{array}{ccc} X & \xleftarrow[\text{stein fact.}]{\tilde{\sigma}} & \widetilde{X} \supset \tilde{\phi}^{-1}(\widetilde{\mathcal{C}}) \\ \Big\downarrow \phi & & \Big\downarrow \tilde{\phi}\colon G\text{-cover} \\ \mathcal{C} \subset \mathbb{P}^2 & \xleftarrow[\text{emb. resol.}]{\sigma} & Y \supset \widetilde{\mathcal{C}} := \sigma^{-1}(\mathcal{C}) \end{array}$$

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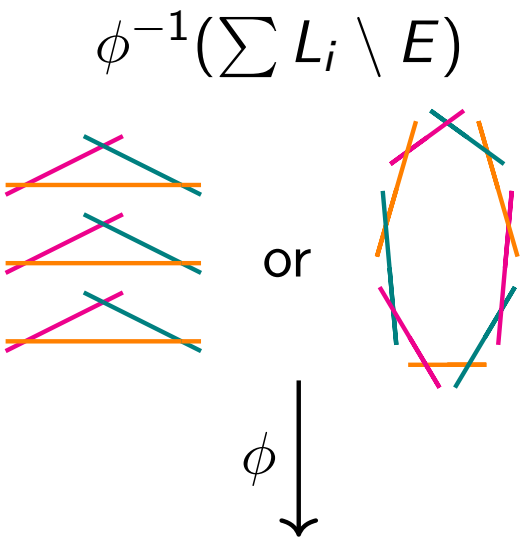
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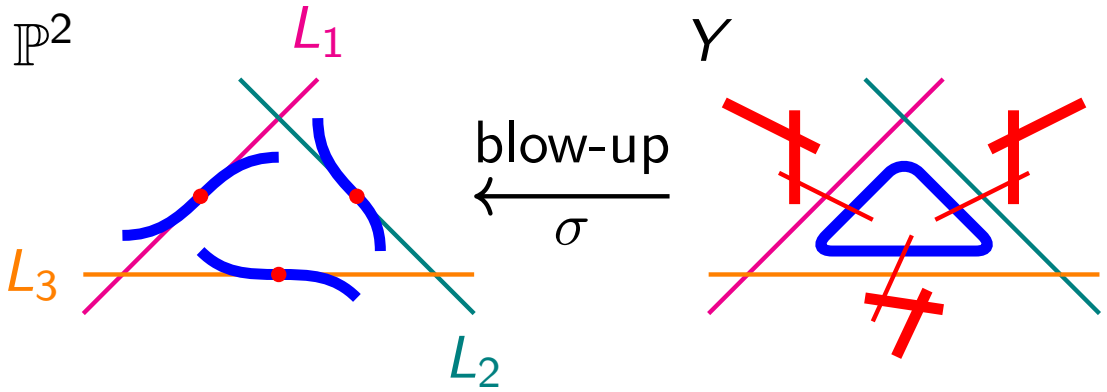
We define the G-combinatorial type $\mathrm{Cmb}_G(\mathcal{C}, \theta)$ of $\mathcal{C}$ for ϕ as the "combinatorial type" of $\tilde{\phi}^{-1}(\widetilde{\mathcal{C}}) \subset \widetilde{X}$ with action of G.

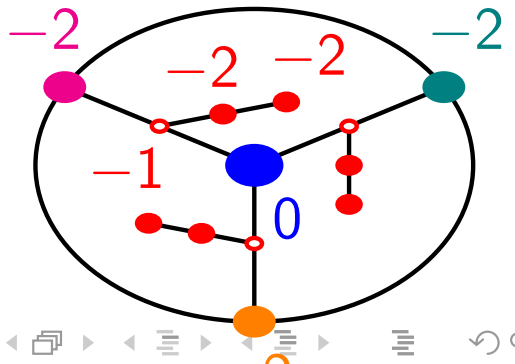
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e.g. (1) $\mathcal{C} = E + L_1 + L_2 + L_3$: an AS curve of type $(3; (3), (3), (3))$.

Let $\phi : X \to \mathbb{P}^2$ be the $\mathbb{Z}/3\mathbb{Z}$-cover branched along E.



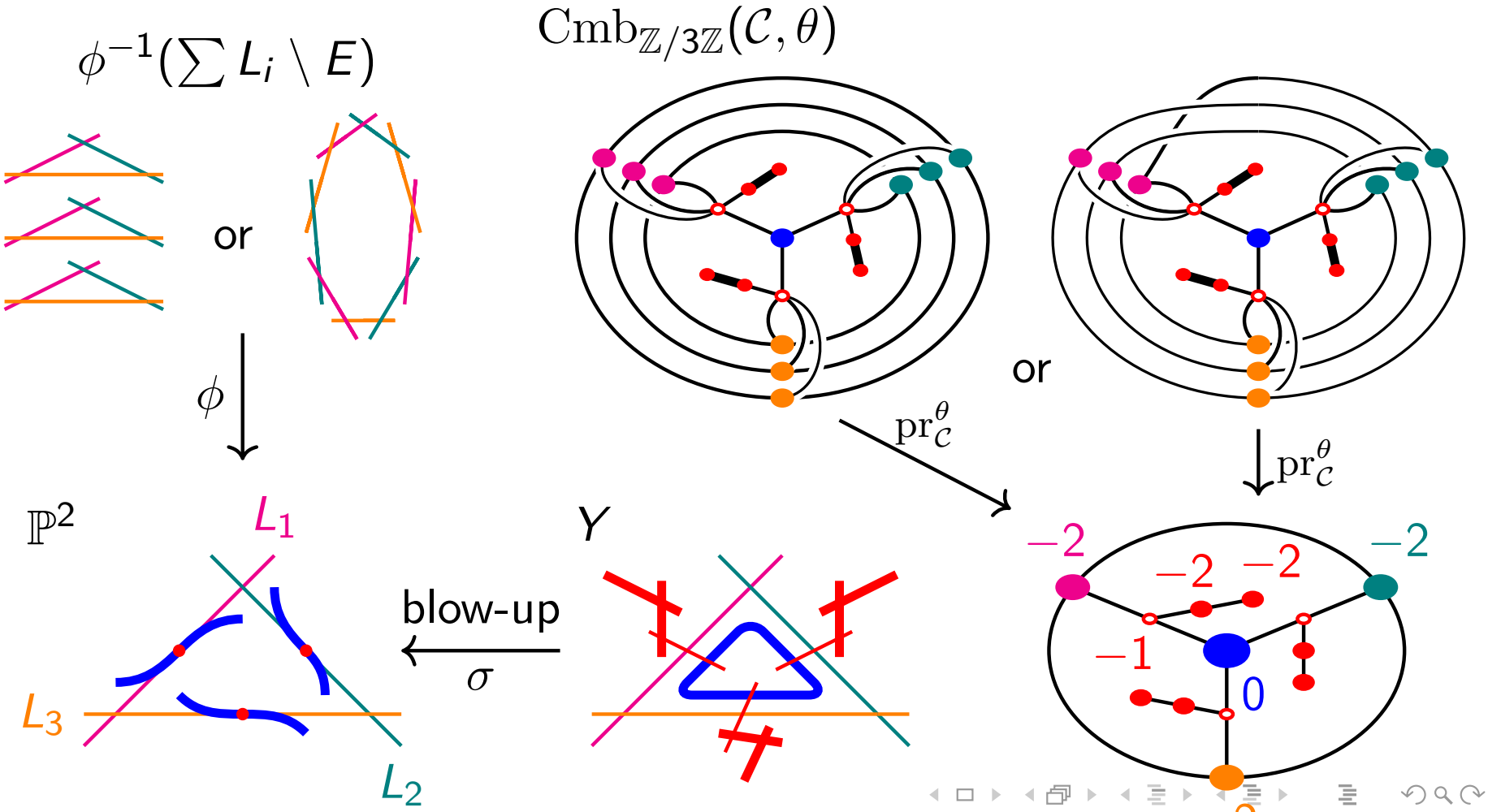




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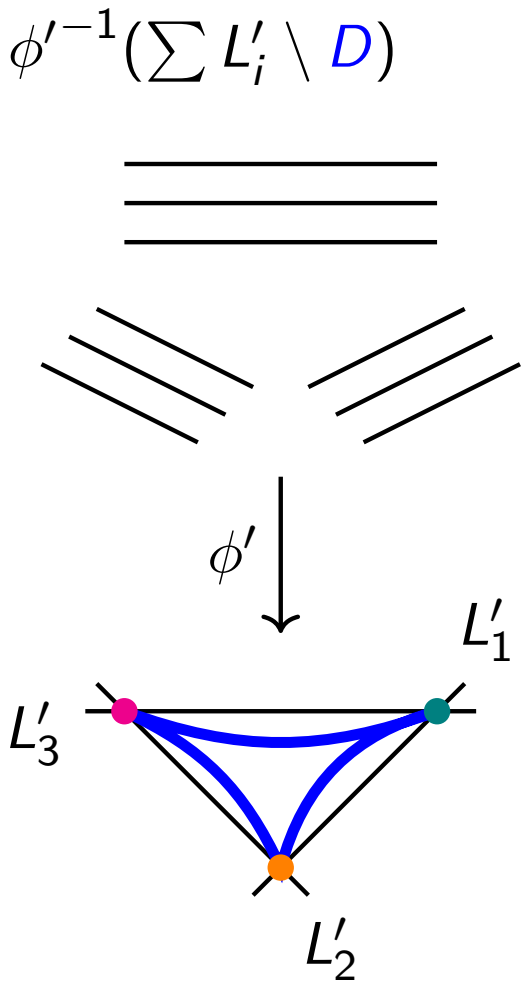


§4 G-combinatorial type 1

(2) $\mathcal{C}' := D + L_1' + L_2' + L_3'$ with D: deg. 6 with three $(3,4)$-cusps.
$\phi' : X' \to \mathbb{P}^2$: the $\mathbb{Z}/3\mathbb{Z}$-cover branched along D. $x^3 + y^4 = 0$

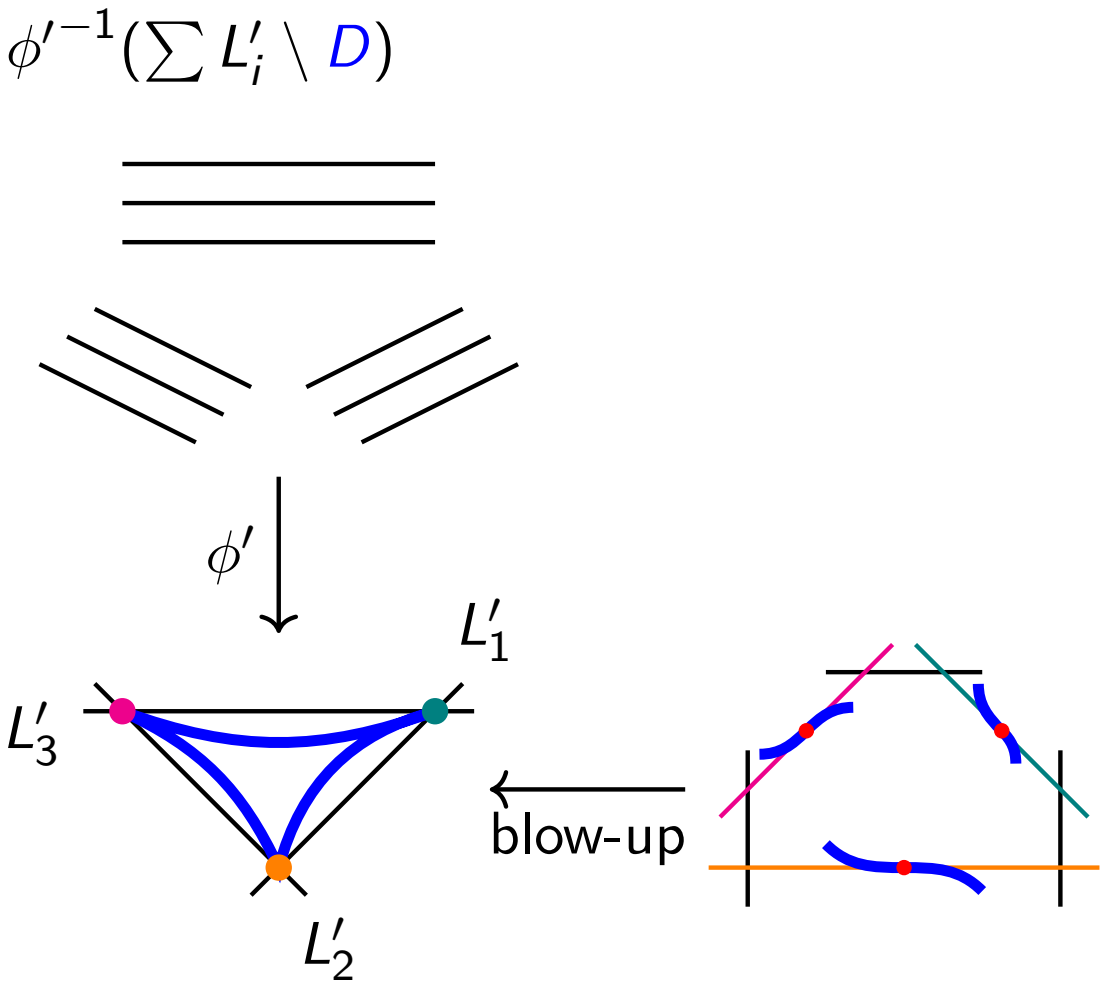
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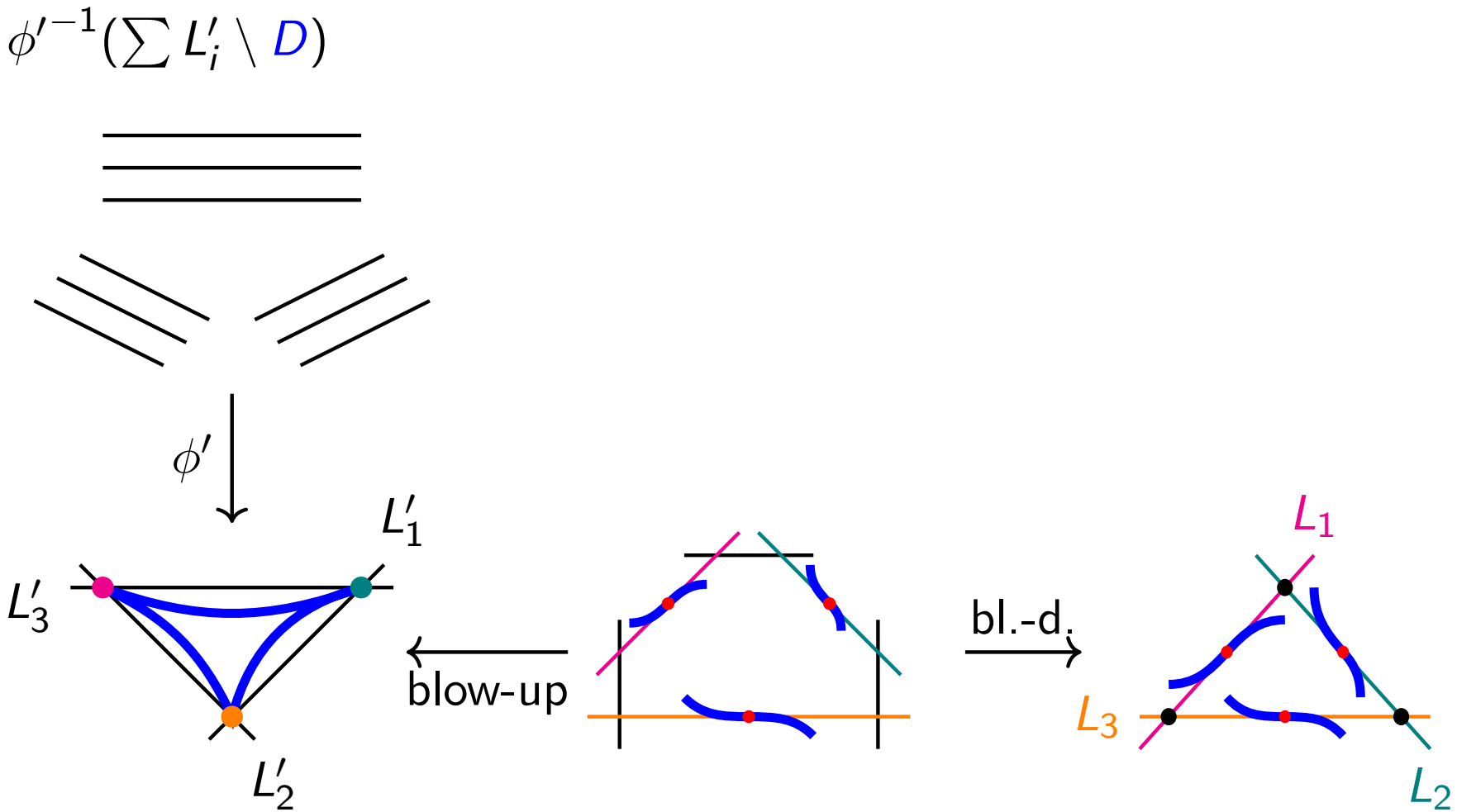
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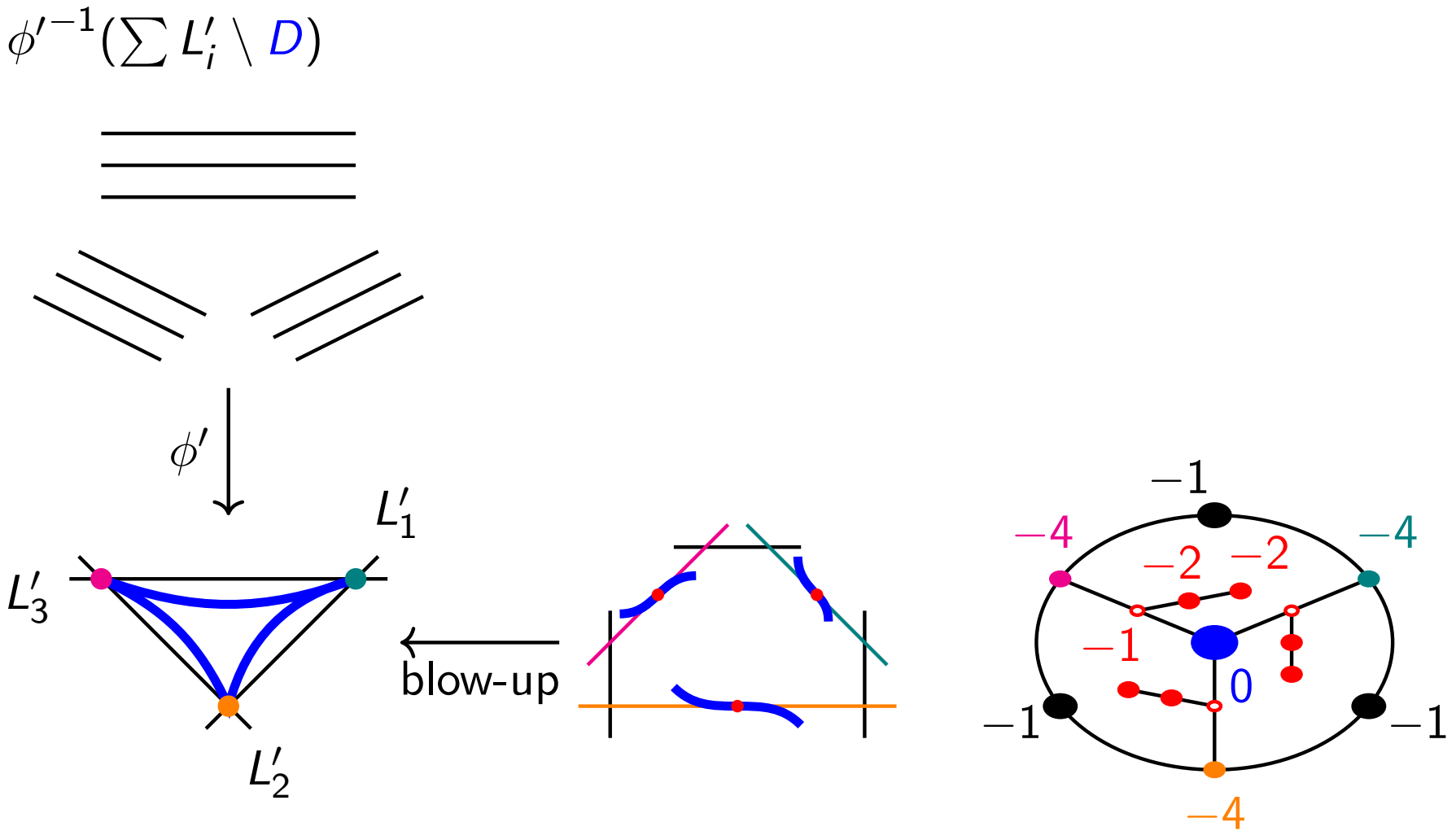
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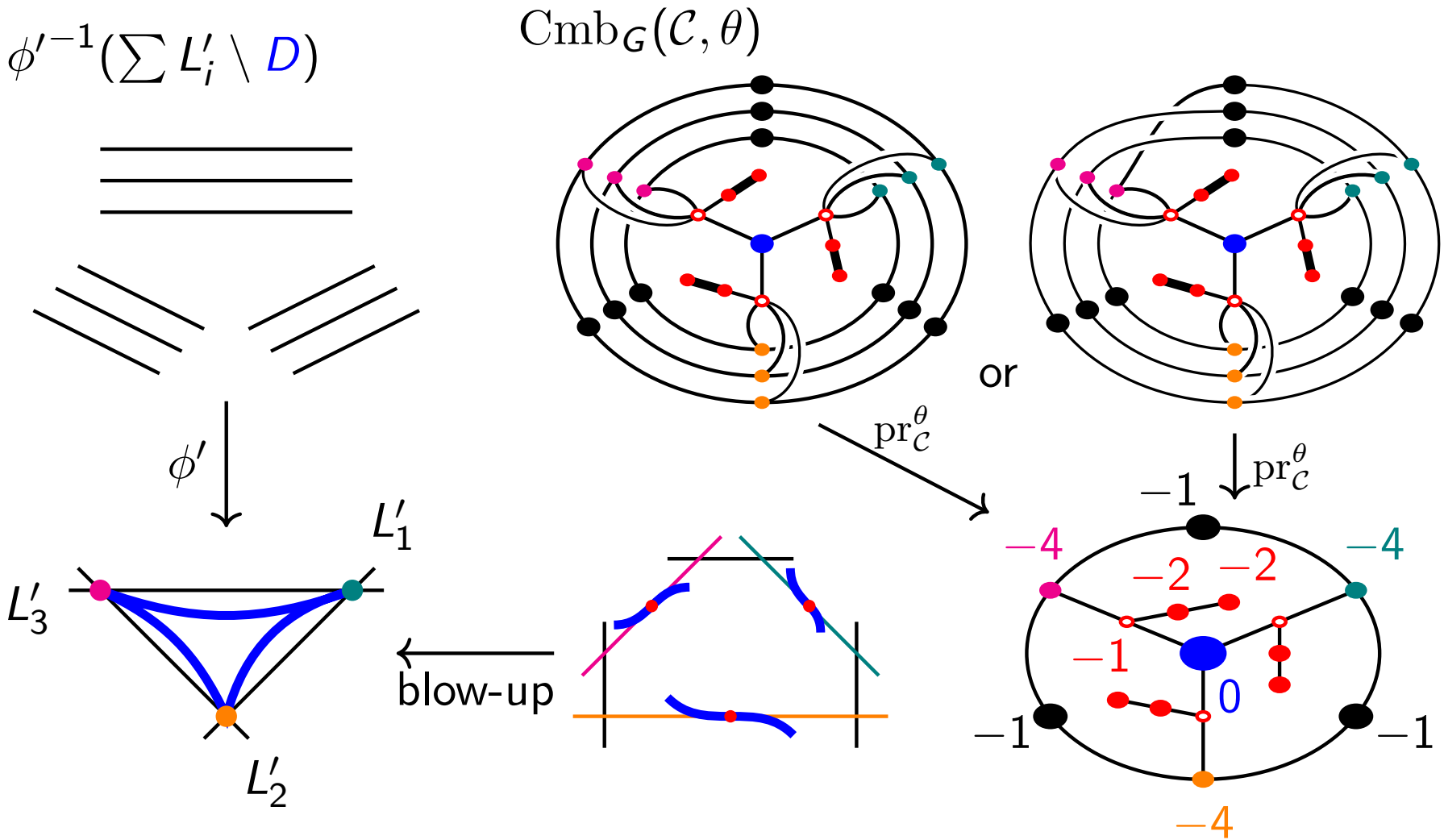
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As the previous example, we can distinguish the embedded topology of the quadratic transformations of AS curves of type $(3; (3), (3), (3))$. This can be generalized as follows:

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Definition (Quasi-triangular curve)

Let $\mathcal{C} \subset \mathbb{P}^2$ be an AS curve of type $(d; \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$.
We call the quadratic transformation $\mathcal{C}' = D + L_1' + L_2' + L_3'$ of $\mathcal{C}$ a *quasi-triangular curve of type* $(2d; \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$.

By the classification of AS curves by splitting invariants, we obtain the following.

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By the classification of AS curves by splitting invariants, we obtain the following.

Theorem (S., 2026)

The embedded topology of quasi-triangular curves is classified by G-combinatorial type.

§5 G-combinatorial type 2

definition and invariance

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Precisely, $\mathrm{Cmb}_G(\mathcal{C},\theta)$ is the 6-tuple $(\Gamma^\theta_{\mathcal{C}}, \boldsymbol{g}^\theta_{\mathcal{C}}, \boldsymbol{e}^\theta_{\mathcal{C}}, \boldsymbol{m}^\theta_{\mathcal{C}}; \rho^\theta_{\mathcal{C}}, \mathrm{pr}^\theta_{\mathcal{C}})$ of

$$\begin{array}{ccc} X & \xleftarrow{\tilde{\sigma}} & \widetilde{X} \supset \tilde{\phi}^{-1}(\widetilde{\mathcal{C}}) \\ \phi \downarrow & & \downarrow \tilde{\phi} \\ \mathcal{C} \subset \mathbb{P}^2 & \xleftarrow{\sigma} & Y \supset \widetilde{\mathcal{C}} := \sigma^{-1}(\mathcal{C}) \end{array}$$

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- the dual graph $\Gamma^\theta_{\mathcal{C}}$ of $\tilde{\phi}^{-1}(\widetilde{\mathcal{C}})$,
- the map $\boldsymbol{g}^\theta_{\mathcal{C}}\colon \mathcal{V}(\Gamma^\theta_{\mathcal{C}}) \to \mathbb{Z}$ of genera,
- the map $\boldsymbol{e}^\theta_{\mathcal{C}}\colon \mathcal{V}(\Gamma^\theta_{\mathcal{C}}) \to \mathbb{Z}$ of "self-intersection numbers",
- the map $\boldsymbol{m}^\theta_{\mathcal{C}}\colon \mathcal{Y}(\Gamma^\theta_{\mathcal{C}}) \to \mathrm{Mat}(2,\mathbb{Z})$ of "covers $T_{\tilde{y}} \to T_y$",
- the action $\rho^\theta_{\mathcal{C}}\colon G \times \Gamma^\theta_{\mathcal{C}} \to \Gamma^\theta_{\mathcal{C}}$ induced from the G-cover $\tilde{\phi}$,
- the projection $\mathrm{pr}^\theta_{\mathcal{C}}\colon \Gamma^\theta_{\mathcal{C}} \to \Gamma_{\mathcal{C}}$, where $\mathrm{Cmb}(\mathcal{C}) = (\Gamma_{\mathcal{C}}, \boldsymbol{e}, \mathrm{Str}_{\mathcal{C}})$.

$$\begin{array}{ccc} X & \xleftarrow{\tilde{\sigma}} & \widetilde{X} \supset \tilde{\phi}^{-1}(\widetilde{\mathcal{C}}) \\ \phi \downarrow & & \downarrow \tilde{\phi} \\ \mathcal{C} \subset \mathbb{P}^2 & \xleftarrow{\sigma} & Y \supset \widetilde{\mathcal{C}} := \sigma^{-1}(\mathcal{C}) \end{array}$$

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Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ be two plane curves with $\theta_i \colon \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_i) \twoheadrightarrow G$.
Let $\phi_i \colon X_i \to \mathbb{P}^2$ be the G-cover induced by θ_i for each $i = 1, 2$.
Write $\mathrm{Cmb}(\mathcal{C}_i) = (\Gamma_i, \boldsymbol{e}_i, \mathrm{Str}_i)$ and

$$\mathrm{Cmb}_G(\mathcal{C}_i, \theta_i) = (\Gamma_i^\theta, \boldsymbol{g}_i^\theta, \boldsymbol{e}_i^\theta, \boldsymbol{m}_i^\theta; \rho_i^\theta, \mathrm{pr}_i^\theta).$$

Theorem (Invariance of G-comb.) (S., 2026)

Assume $^\exists h \colon \mathbb{P}^2 \to \mathbb{P}^2$*: homeo. and* $^\exists \tau \colon G \to G$*: auto.*
s.t. $h(\mathcal{C}_1) = \mathcal{C}_2$ *and* $\tau \circ \theta_1 = \theta_2 \circ h_*$.
Let $\varphi \colon \mathrm{Cmb}(\mathcal{C}_1) \to \mathrm{Cmb}(\mathcal{C}_2)$ *be the equiv. map induced by* h.

Then $^\exists \varphi_G^\theta \colon \mathrm{Cmb}_G(\mathcal{C}_1, \theta_1) \to \mathrm{Cmb}_G(\mathcal{C}_2, \theta_2)$*: a G-equiv. map w.r.t. τ and φ.*

Here a G-equiv. map φ_G^θ w.r.t. τ and φ is
an isom. $\varphi_G^\theta \colon \Gamma_1^\theta \to \Gamma_2^\theta$ compatible with $\boldsymbol{g}_i^\theta$, $\boldsymbol{e}_i^\theta$, $\boldsymbol{m}_i^\theta$, pr_i^θ, φ and
$\varphi_G^\theta(\rho_1^\theta(g, w_1)) = \rho_2^\theta(\tau(g), \varphi_G^\theta(w_1))$ for $^\forall g \in G$, $^\forall w_1 \in \Gamma_1^\theta$.

Thank you very much!!