Singularities in affine reflection and symetrization of convex plane curves

Łojasiewicz 100 Conference

Joint work with Peter Giblin and Michał Zwierzyński

Polish-Japanese Singularity Theory Working Days

Professor Łojasiewicz





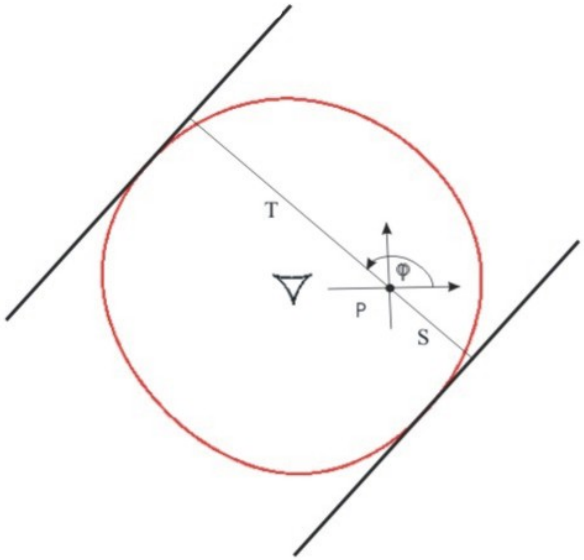
Polish-Japanese Singularity Theory Working Days

Professor Łojasiewicz and Professor Hironaka



Center of Symmetry - Center Symmetry Set (CSS)

Affinly invariant construction

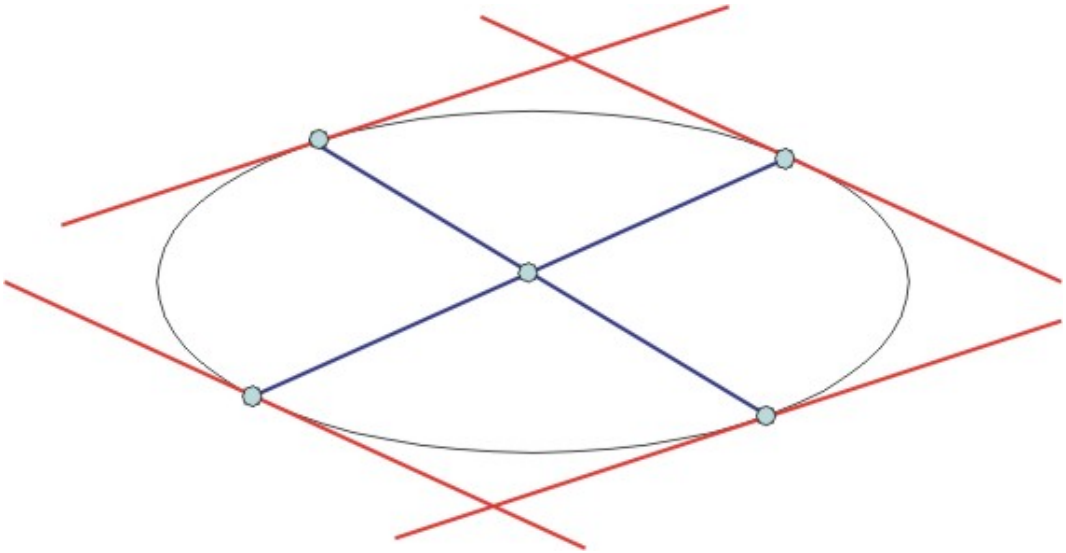


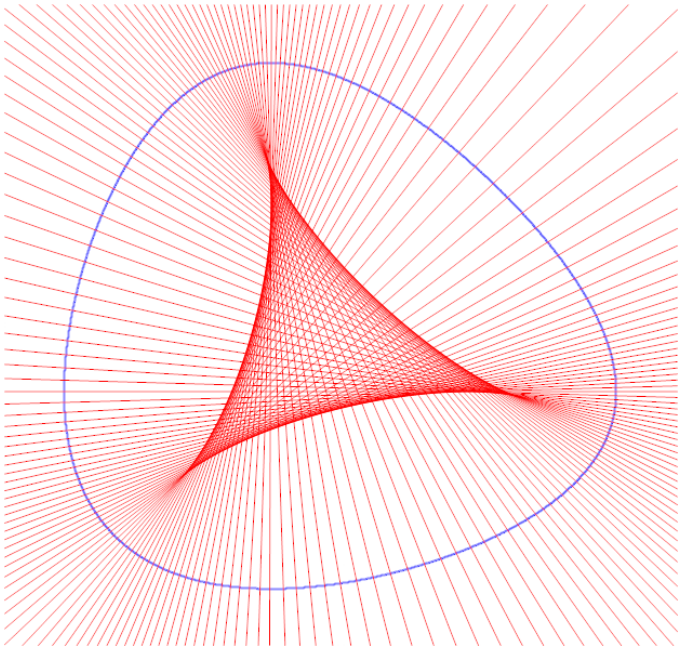
$$F(p;\varphi) = \frac{t}{s}$$

$$\Sigma = \{p \in R^2; exists\ \varphi\ such\ that\ \frac{\partial F}{\partial \varphi}(p,\varphi) = 0\ and\ \frac{\partial^2 F}{\partial \varphi \partial \varphi}(p,\varphi) = 0\}$$

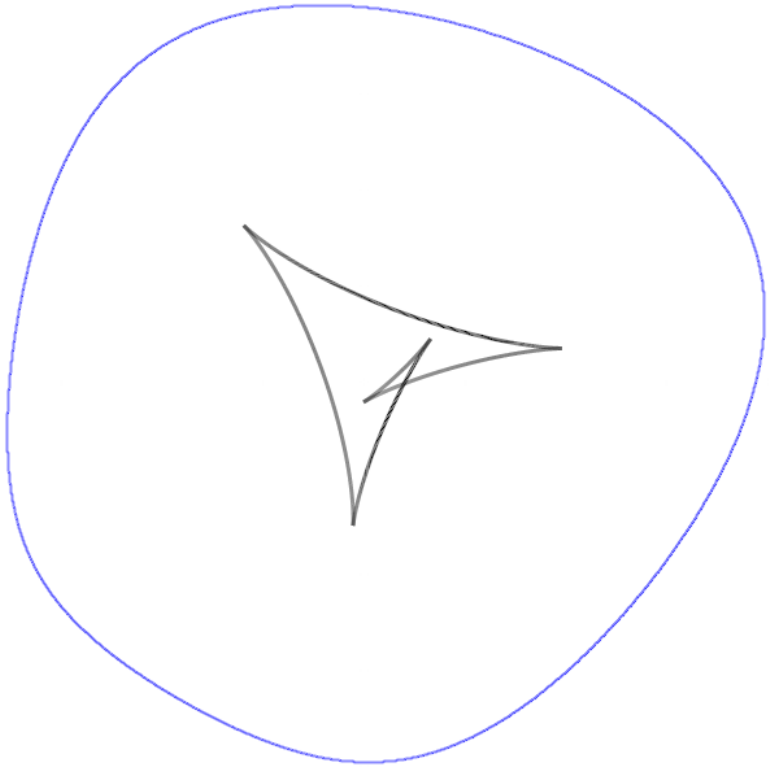
Another construction of the Center Symmetry Set

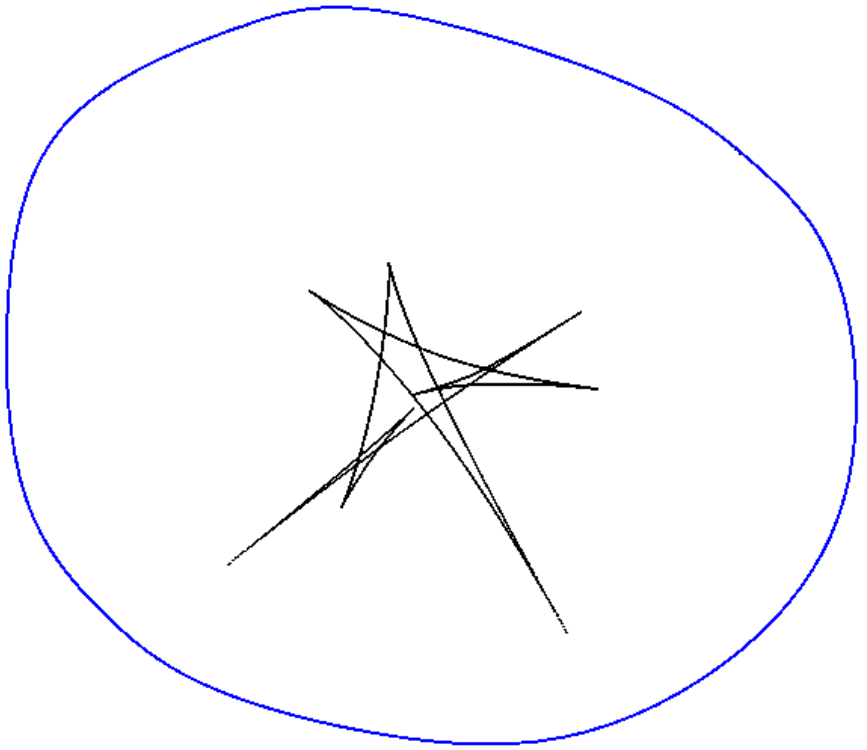
Envelope of tangent chords





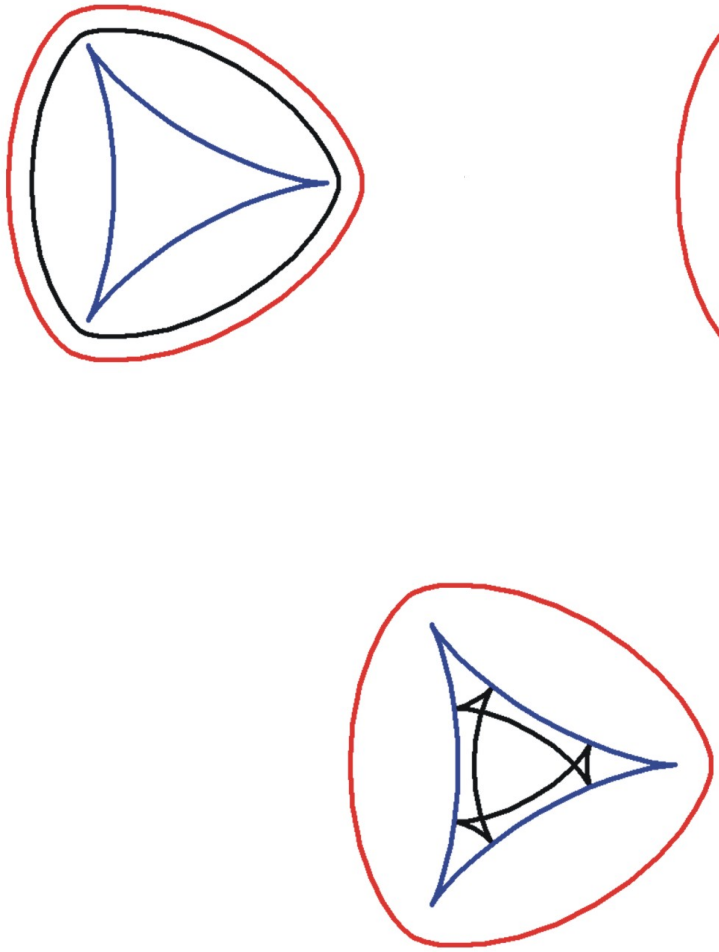
Examples of CSS





Dual Wigner-Berry caustic

Family of affine equidistants

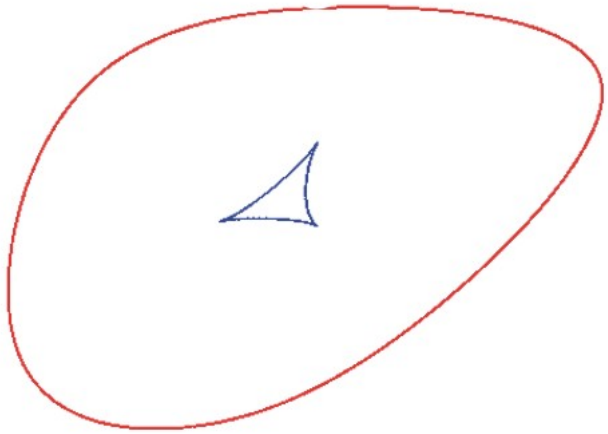


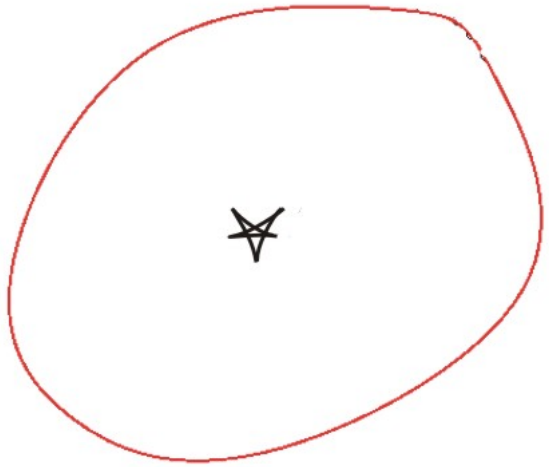


Wigner caustic $W_{\frac{1}{2}}$(right picture) as a dual to CSS

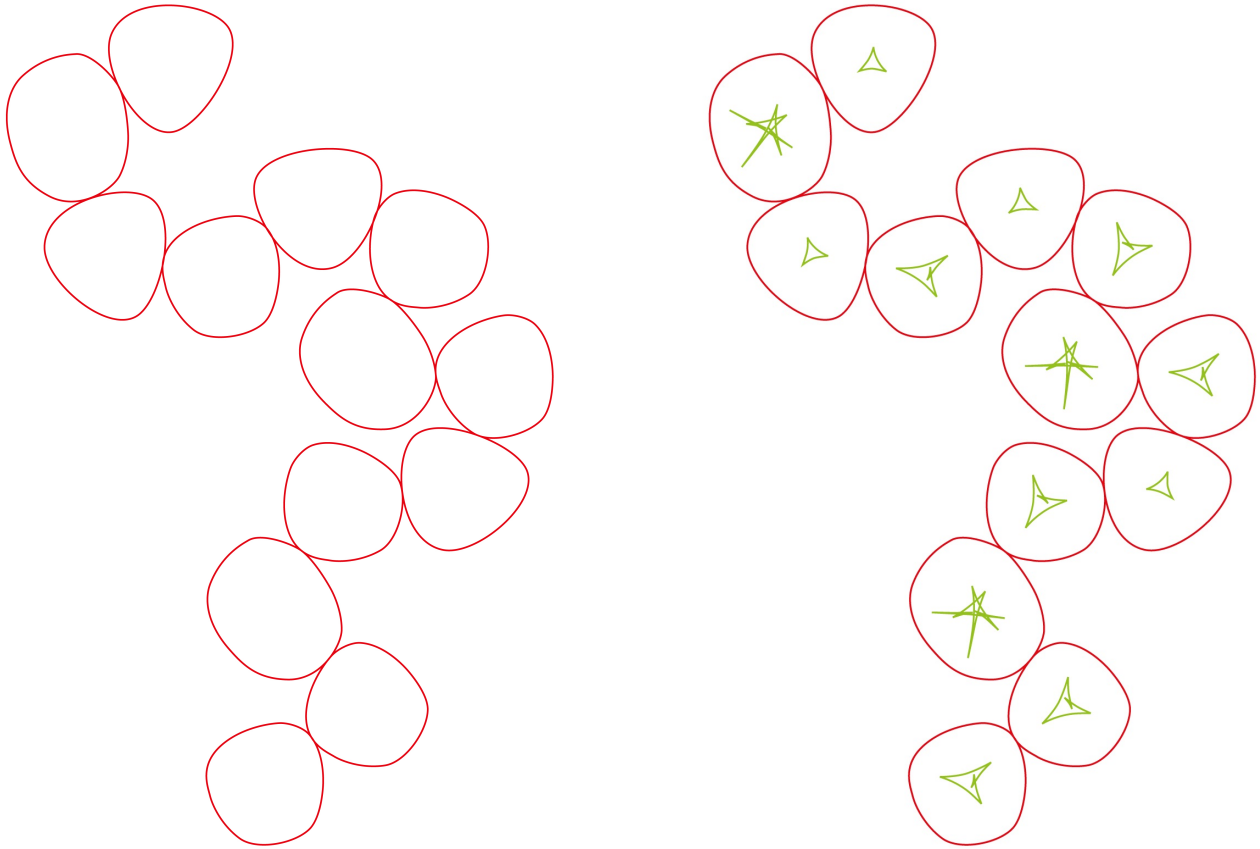
$$W_\lambda = \{\lambda a + (1-\lambda)b \in R^2 : a, b \; points \; of \; parallel \; pair\}$$

Some examples of Wigner caustics



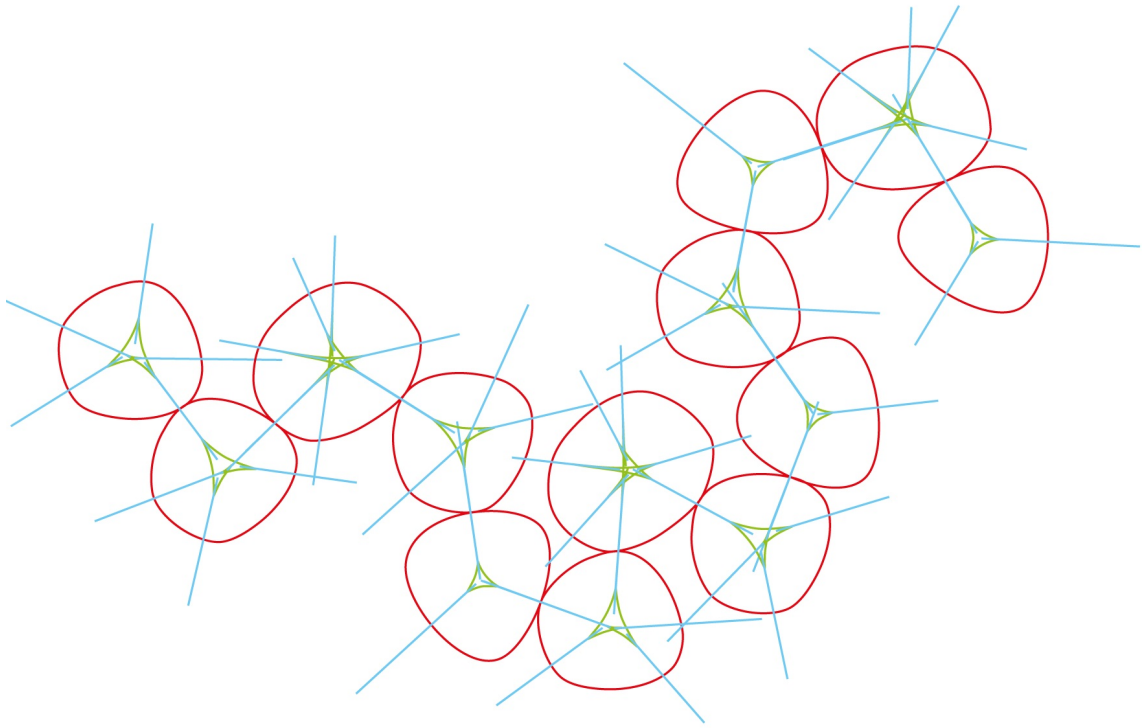


Organization of molecules



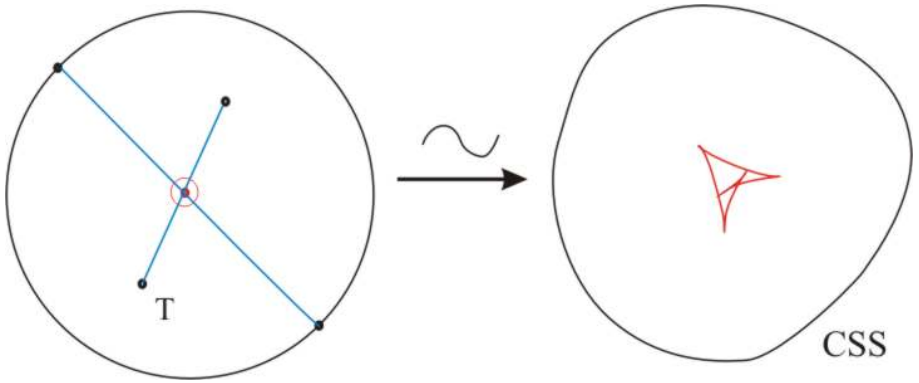
Organization of molecules

through the affine singularities of cell edge

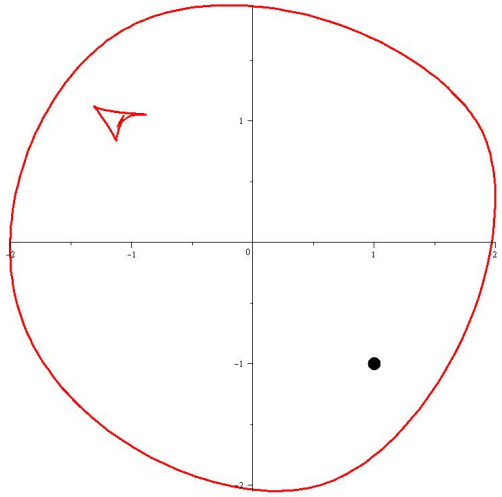


Spaces with symmetry

Deformation and symmetrization

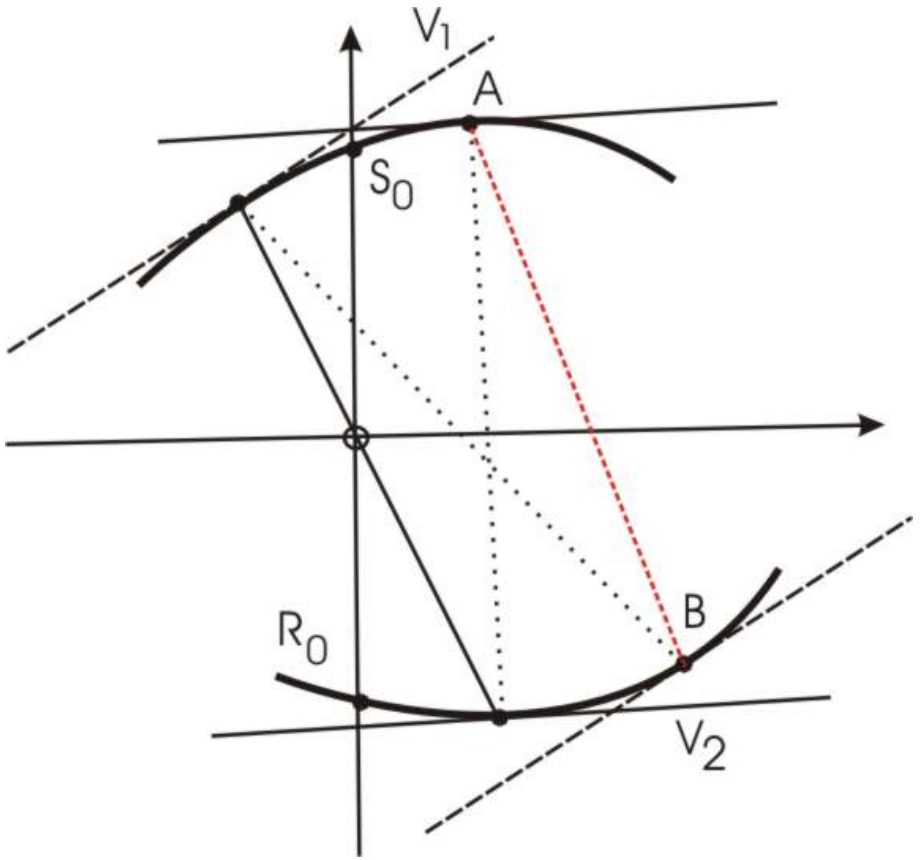


Affine reflection in ovals

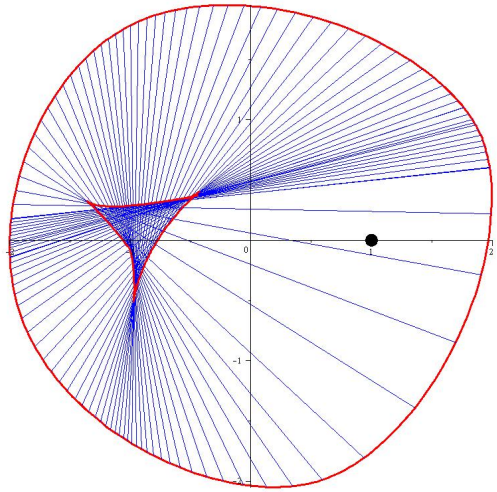


Affine reflection

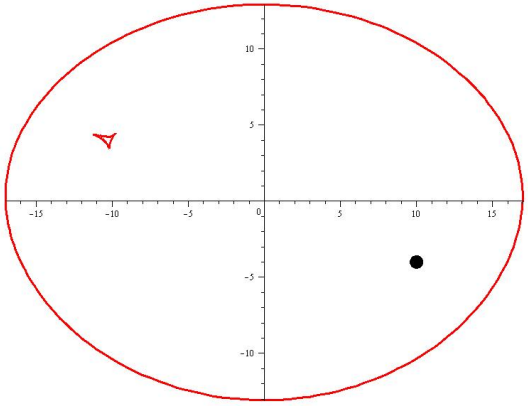
Parallel pairs



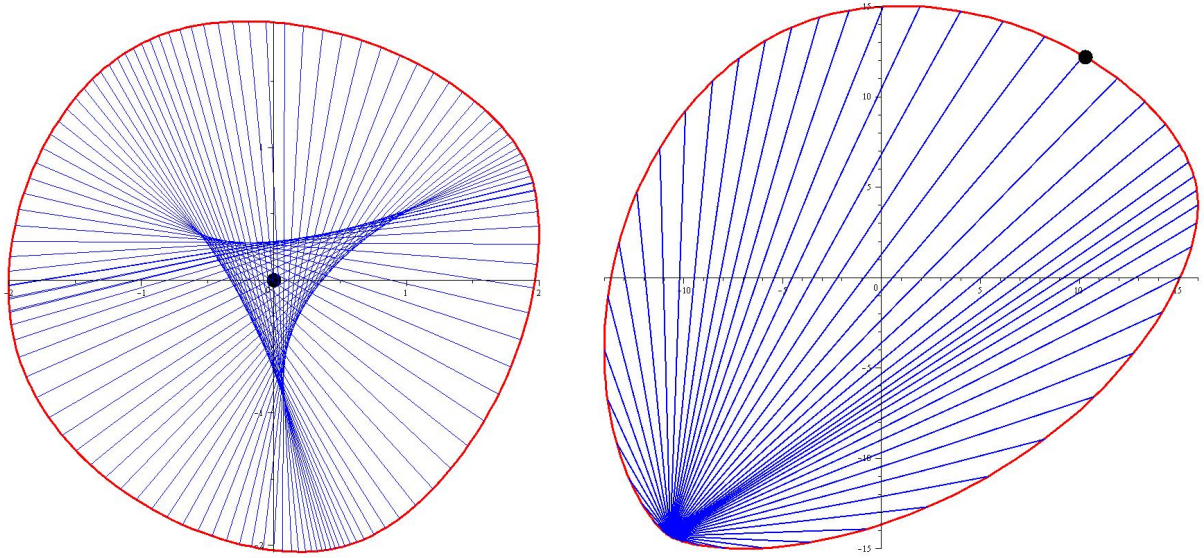
Affine reflection of a point source



Slightly deformed ellipse

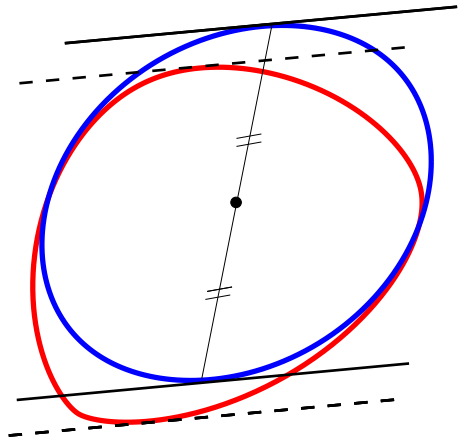


Some examples

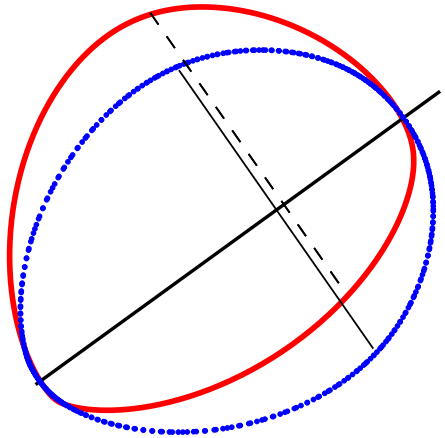


Several ways of symmetrization

The parallel tangent transform

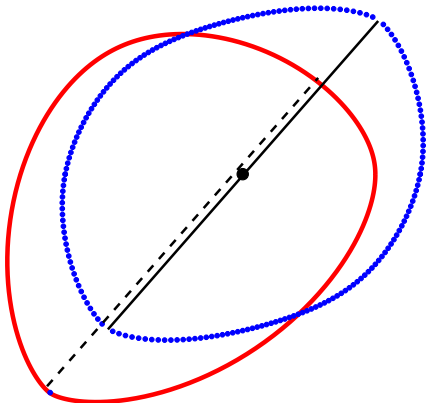


The chords peprpendicular to the line transform

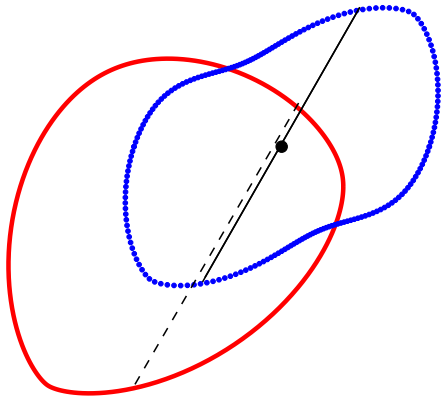


Center Symmetry Transform (CST)

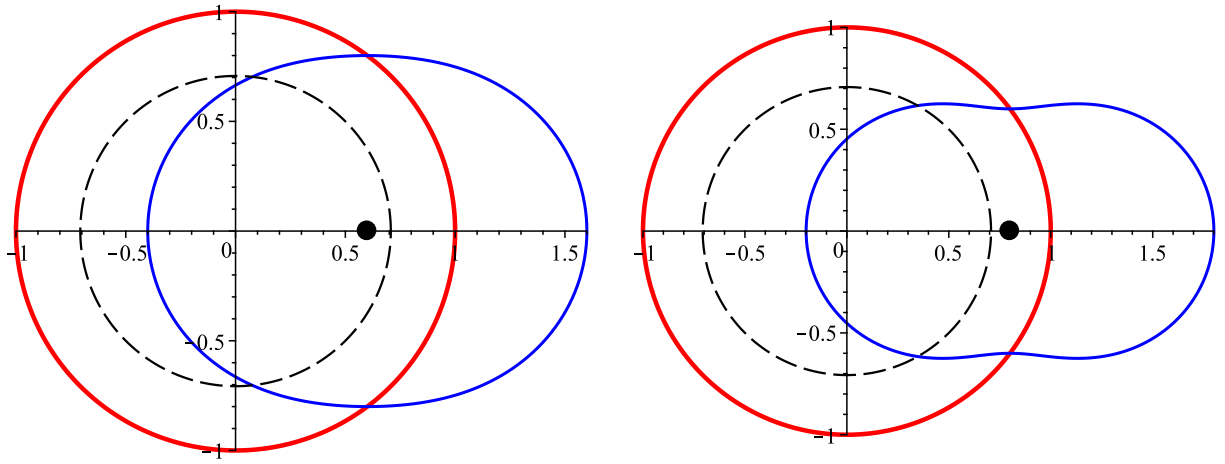
Point inside the convex curve



Nonconvex CST



CST of a circle or ellipse



Proposition Let Γ be an ellipse, and R the region inside Γ consisting such P that the CST of Γ wrt P is strictly convex. Then R is the interior of a concentric ellipse, the scale factor $\frac{1}{\sqrt{2}}$. It is a global CCST boundary of γ.

If $P = (p, 0)$ then the CST:

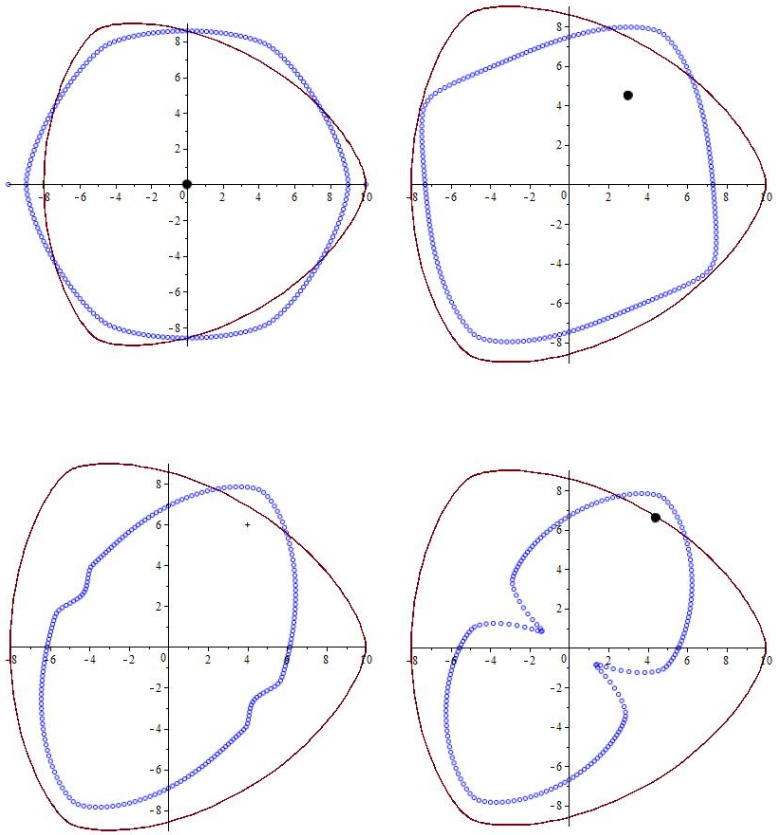
$$(\sqrt{1 - p^2 sin^2 t} cost, \sqrt{1 - p^2 sin^2 t} sint)$$

and the inflection points:

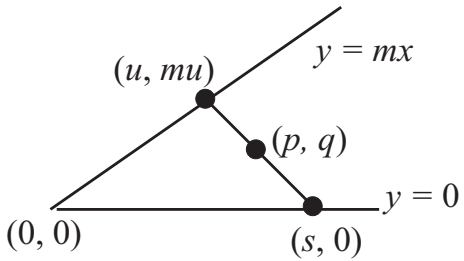
$$sin^2 t = \frac{1 + p^2}{2p^2(2 - p^2)}$$

The condition is $p \geq \frac{1}{\sqrt{2}}$

Examples of CST of the constant width curve



CST of two intersecting stright lines, triangle and rounded triangle



Equation of collinearity

$$(u-p)q = (mu-q)(p-s)$$

And the formula for CST

$$(x,y) = (\frac{1}{2}(u-s)+p, \frac{1}{2}mu+q)$$

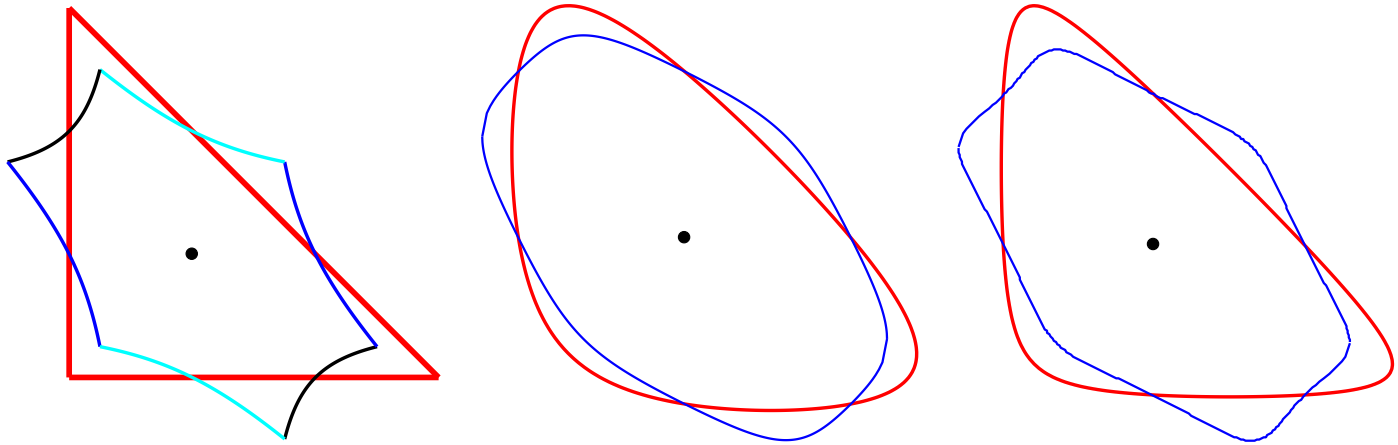
Get the equation of CST, the hyperbola through (p,q):

$$2mxy - 2y^2 - 3mqx + (4q-mp)y + 2mpq - 2q^2 = 0$$

Triangle and rounded triangle with empty convex CST region

Sides: $x = 0, y = 0, x + y = 1, \quad (p, q) = (\frac{1}{3}, \frac{1}{3})$

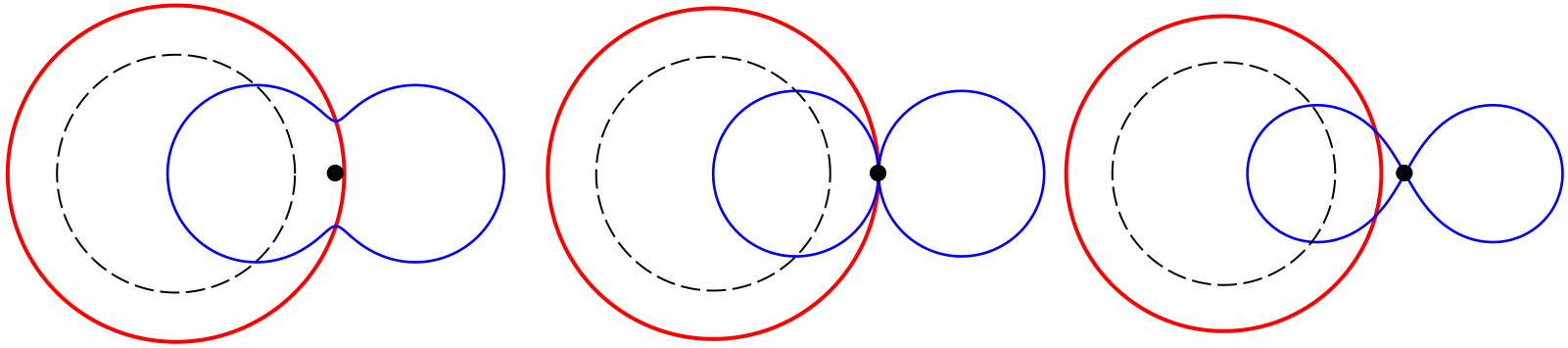
The curves: $xy(x + y - 1) = k$



second $k = \frac{1}{50}$ possibly nonempty convex CST

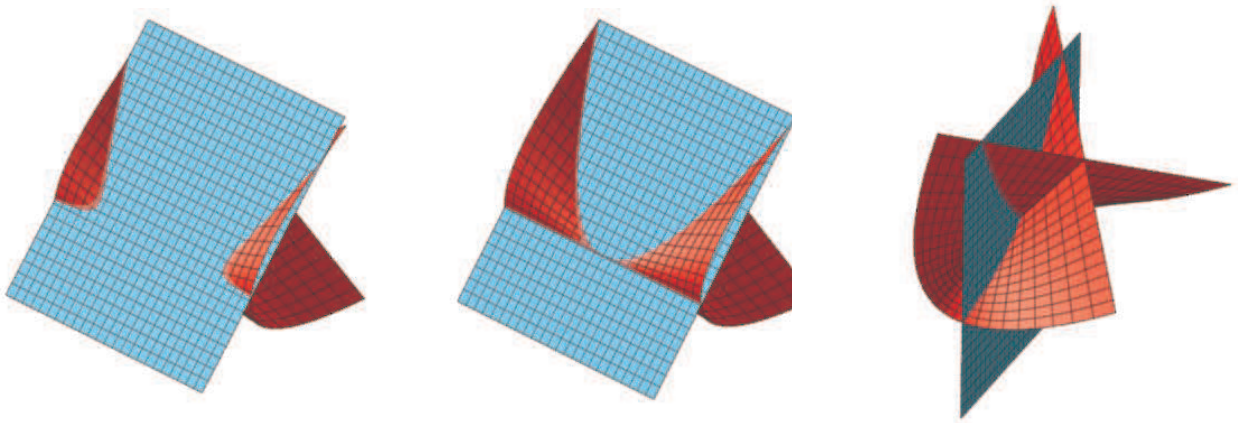
$k = 0, k = \frac{1}{130}$ empty convex CST

CST for basepoints approaching the curve



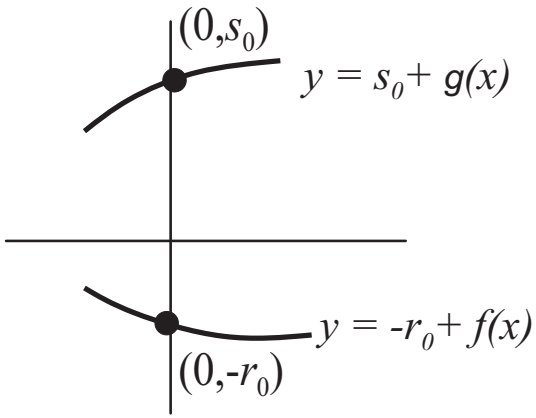
Proposition If P moves across C without an inflection, CST changes from two smooth branches with two inflections each (P inside C) through two smooth branches with ordinary tangency (P on C) to a crossing with each branch having an inflection (P outside C). In particular for a smooth closed strictly convex curve C and a basepoint P sufficiently close to and inside C the CST is never convex.

Proposition Sections of a standard Whitney umbrella (crosscap) correspond with the local evolution of a CST as the basepoint crosses the curve Γ.



Local evolution of CST - sections of a standard Whitney umbrella.

Conditions for higher inflections on the CST



Lower curve: $(u, -r_0 + f_1 u + f_2 u^2 + f_3 u^3 + \ldots), r_0 > 0.$

Upper curve: $(x, s_0 + g_1 x + g_2 x^2 + g_3 x^3 + \ldots), s_0 > 0.$

Collinearity with the origin: $u(s_0 + g(x)) = x(-r_0 + f(u)).$

Solve this equation according to u : $u = U(x)$ and substitute into f,

$$F(x) = f(U(x))$$

The parametrization of one side of CST

Translated to the origin

$$(X(x), Y(x)) = (\frac{1}{2}(x - U(x)), \frac{1}{2}(r_0 - F(x) + s_0 + g(x)))$$

The conditions for higher inflection on the CST at $x = 0$

$$X'Y'' - X''Y' = 0 \quad X'Y''' - X'''Y' = 0$$

Equivalent at $x = 0$ to:

$$r_0 s_0 (f_1 - g_1)^2 = (r_0 + s_0)(f_2 r_0^2 - g_2 s_0^2) \tag{1}$$

$$(r_0 + s_0)(f_3 r_0^3 + g_3 s_0^3) = (f_1 - g_1)(f_2 r_0^3 + 2 f_2 r_0^2 s_0 + 2 g_2 r_0 s_0^2 + g_2 s_0^3) \tag{2}$$

A *double inflection chord*, relative to a given basepoint, is a chord for which the endpoints occur at degenerate inflexions on the CST. Usually of order 2, but could be 3 in the case of a singular CCST boundary (CCST - Convex Center Symmetry Transforms)

Local structure of the CCST boundary

Proposition Suppose that the conditions (1) and (2) hold so that the chord joining $(0, -r_0)$ and $(0, s_0)$ is a double-inflection chord for the CST relative to $(0, 0)$. Then the chord is tangent to the local CCST boundary if $f_2 r_0^2 + g_2 s_0^2 = 0$.

Singular point of the CCST boundary

The additional condition that the inflection is *triple*:

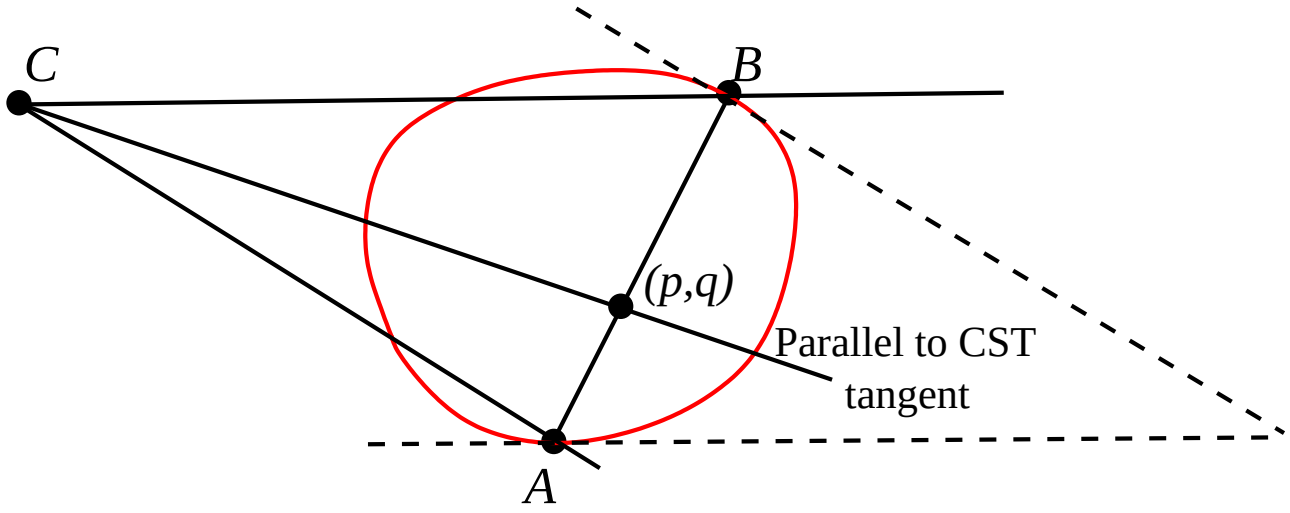
$$X' Y^{(4)} - X^{(4)} Y' = 0 \tag{3}$$

The cusp singularity example fulfilling (1), (2), (3):

$$r_0 = 2, s_0 = 5, f(u) = u^2, g(x) = 2x - \frac{12}{175}x^2 - \frac{456}{6125}x^3 + \frac{81584}{1071875}x^4$$

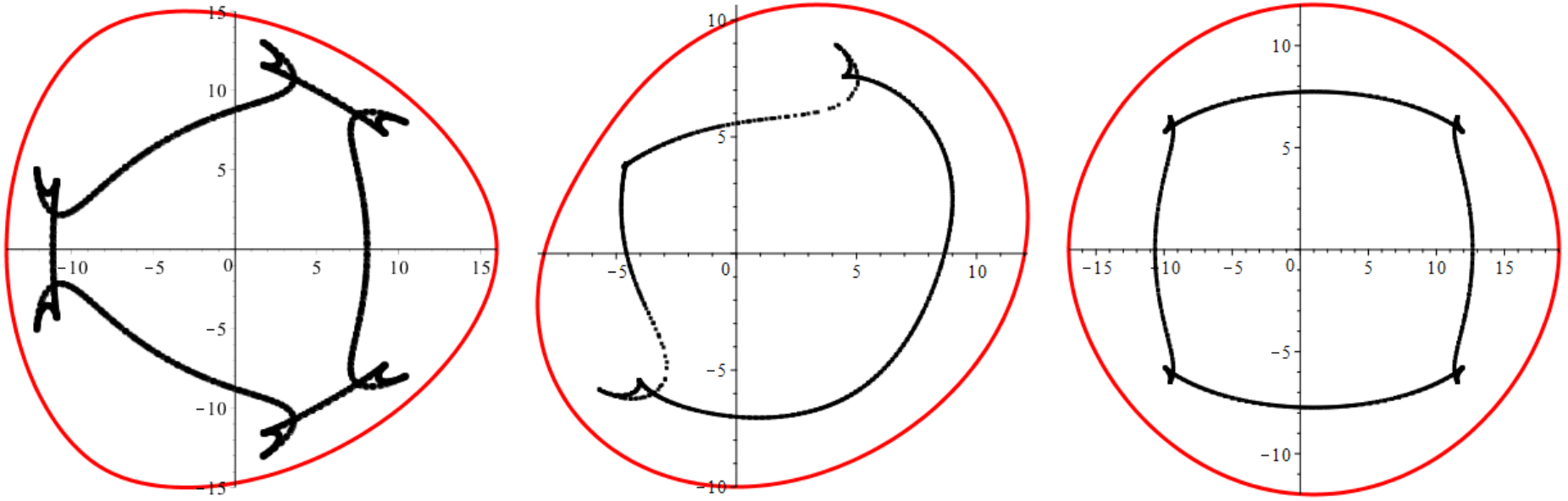
The CST tangents

The "parallelogram construction"

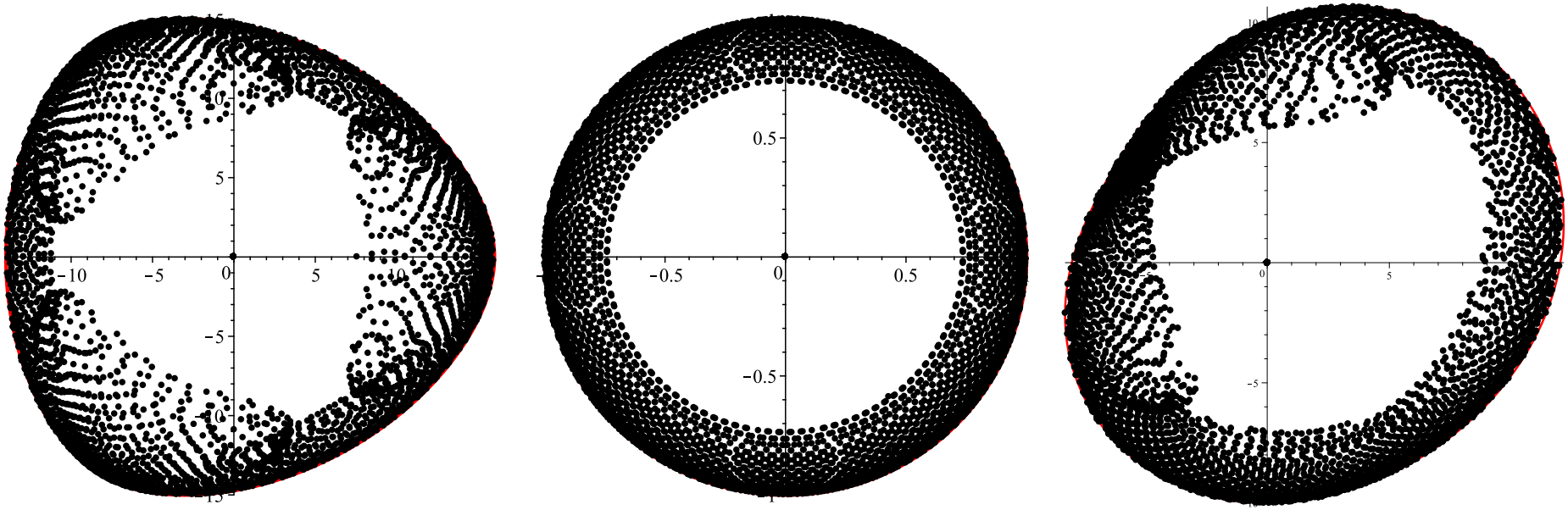


Proposition The dashed lines are tangents to the given curve at the points A, B of the chord and (p, q) is the base point. Lines are drown through A parallel to the tangen at B and vice versa. These meet at C and the line joining C to the basepoint is parallel to the tangent lines to the CST at either of the corresponding points.

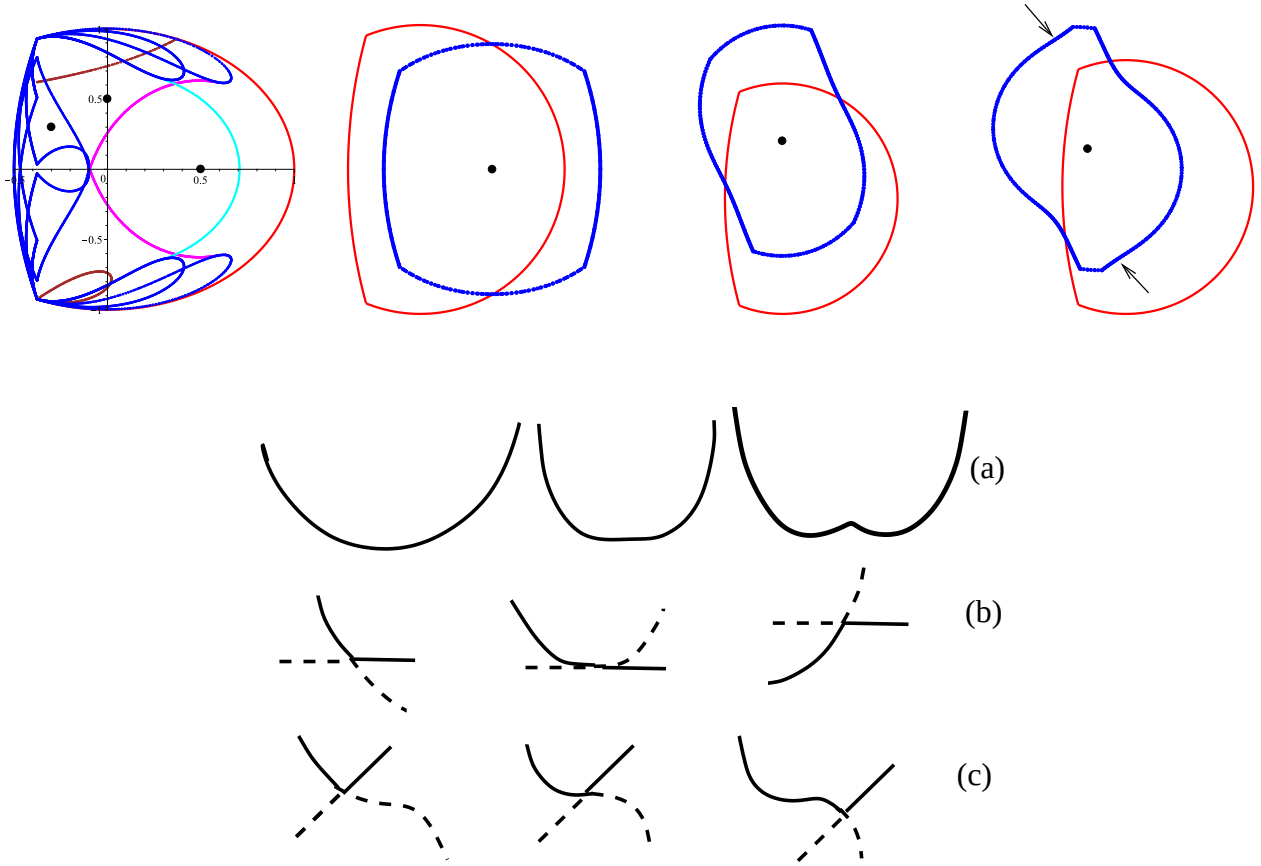
The structure of the CCST boundary



Non convex region of CST



A piecewise smooth example



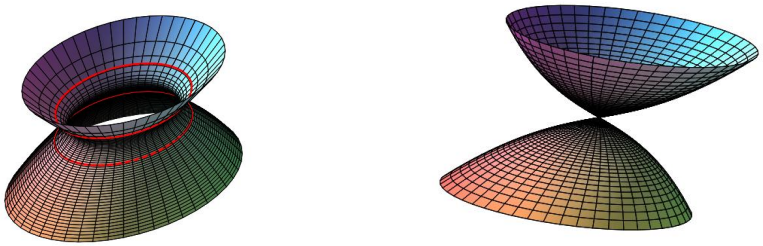
a) Double inflection transition

b) Two intersecting branches of CST become tangent

c) An inflection on one branch is crossed by another branch.

CSTs of surfaces in R^n

Paraboloid from inside and outside



CST of a unit sphere

