Yomdin–Gromov parametrizations over valued fields with analytic structure

Conference Łojasiewicz 100

Analytic Geometry and Singularity Theory

Kraków, 21 September 2026

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Main aim

My talk ([N5]) concerns uniform Yomdin–Gromov parametrizations ([Yo-1, Yo-2, Gro]) together with an estimate of their number. I generalize to arbitrary equicharacteristic zero valued fields with analytic structure partial results by Cluckers–Comte–Forey–Loeser ([CCL, CFL]) achieved, among others, over equicharacteristic zero, complete, discretely valued fields, possibly augmented by adding certain restricted analytic function symbols.

To this end, I establish a certain strong analytic stratification of definable sets, based on a definable non-Archimedean version of Bierstone–Milman's [BM] canonical desingularization algorithm from my earlier paper [N2].

Other basic tools are: elimination of valued field quantifiers, term description of functions definable in analytic structures, and Lipschitz cell decomposition compatible with RV-parametrized sets in Hensel minimal structures.

Philosophy behind the proof

The approach of [CCL, CFL] consists of first obtaining a certain strong T_1-parameterization, and subsequently deriving a T_r-parameterization via precomposing by the r-th power functions.

Such a parametrization is defined on an open cell $P \subset K^m$, with zero centers. Moreover, it is globally analytic on each box $B \subset K^m$ of P, and even on each associated box $B_{as} \subset (K^{alg})^m$.

Actually, the passage to the algebraic closure enables the use of Cauchy's estimates, which is crucial for the calculations related to power substitution. The philosophy here is as follows:

- firstly, precomposing a T_1-parametrization with the r-th power functions yields T_r-property on boxes;
- secondly, global T_1-property and T_r-property on boxes imply global T_r-property.

Real versus non-Archimedean

The idea of composing with r-th power functions was earlier applied by Cluckers–Pila–Wilkie [CPW] in real subanalytic geometry. There the difficulty lies in that o-minimal cells have frontiers which are also modified when composed with power functions.

On the other hand, the absence of frontiers and of convexity arguments in non-Archimedean geometry was a challenge leading Cluckers–Comte–Forey–Loeser to the concept of T_r-maps.

Generally, I adopt their strategy in the general settings of separated analytic structures. The languages used:

$\mathcal{L}_{\mathcal{A}}$ — the primary analytic language from [N4];
$\mathcal{L}^*_{\mathcal{A}}$ — the expansion of $\mathcal{L}_{\mathcal{A}}$ by adding the Henselian functions (to obtain a term structure of definable functions);
$\mathcal{L}^\dagger_{\mathcal{A}}$ — a very limited, algebraic Skolemization of $\mathcal{L}^*_{\mathcal{A}}$ in the sort RV (to obtain an enhanced description of definable functions).

The strategy

I also need a term description of definable functions after a very limited algebraic Skolemization in the sort RV.

Because of Cauchy's estimates when substituting power functions, I separately treat the case where the value group is densely ordered and the opposite one, which is much more delicate.

In the latter case, the challenge for me was the passage to the algebraic closure of the ground field K in the general context of valued fields with analytic structure. This equivalent of the complexification in real algebraic geometry was developed based on the technique of certain strong stratifications.

This technique, in turn, was achieved via elimination of valued field quantifiers and desingularization of terms by blowing up. The latter result relies on a definable non-Archimedean version of Birstone–Milman's canonical desingularization algorithm from my paper [N2].

Desingularization of terms

Theorem

Consider a finite number of $\mathcal{L}_{\mathcal{A}}$-terms $t_1(x), \ldots, t_p(x)$ on $(K^\circ)^n$. Then $(K^\circ)^n$ is a finite disjoint union of strong analytic, subanalytic manifolds $S_k := \sigma_k(V_k)$, $k = 1, \ldots, s$, such that:
1) Each $\sigma_k : N_k \to (K^\circ)^n$ is a multi-blowup with exceptional divisor E_k (from a certain transformation to normal crossings).
2) Each $U_k \subset N_k$ is a clopen T-subdomain of N_k or of the pre-image of an admissible smooth center C, and $V_k := U_k \setminus E_k$.
3) Finally, each pull-back

$$t_i^{\sigma_k} = t_i \circ \sigma_k, \quad i = 1, \ldots, p, \ k = 1, \ldots, s$$

is the restriction to V_k of either a T-analytic function $\omega_{i,k}$ on U_k or its multiplicative inverse. In the latter case, $\omega_{i,k}$ vanishes nowhere on V_k.

Strong stratification

Theorem

Every $\mathcal{L}_{\mathcal{A}}$-definable subset X of $(K^\circ)^n$ is a finite disjoint union of strong analytic manifolds $S_k := \sigma_k(W_k)$, $k = 1, \ldots, s$, such that:
1) Each $\sigma_k : N_k \to (K^\circ)^n$ is a multi-blowup with exceptional divisor E_k; one can assume that N_k are analytic submanifolds of $(K^\circ)^n \times \mathbb{P}^N(K)$ for all k and an integer N large enough.
2) For each $k = 1, \ldots, s$, $W_k := (Z_k \cap \Omega_k) \setminus E_k$, where $U_k \subset N_k$ is a clopen T-subdomain of N_k or of the pre-image of an admissible smooth center C (where a new process starts via induction), Z_k is a T-analytic subset of U_k and Ω_k is an open definable subset of U_k.
3) The restriction $\sigma_k|W_k : W_k \to S_k$ is a homeomorphic parametrization of the stratum S_k. □

It seems that the only results on stratification known before are the ones for subanalytic subsets of affine spaces over an algebraically closed valued field, provided in [LR1, LR2, CL].

Term description after some algebraic Skolemization

Proposition

The theory $T_{\mathcal{A}}$ has the following property uniformly in its models K. Let $f : X \to K$ be an $\mathcal{L}^{\dagger}_{\mathcal{A}}$-definable function on a subset X of K^n. Then f is given piecewise by $\mathcal{L}^{\dagger}_{\mathcal{A}}$-terms, i.e. there exist a finite $\mathcal{L}^{\dagger}_{\mathcal{A}}$-definable partition $X = X_1 \cup \ldots \cup X_s$ and $\mathcal{L}^{\dagger}_{\mathcal{A}}$-terms $t_1, \ldots, t_s$ such that $f(x) = t_i(x)$ for all $x \in X_i$, $i = 1, \ldots, s$.

Theorem

For every $\mathcal{L}^{\dagger}_{\mathcal{A}}$-definable set $X \subset K^n$, there exists a finite decomposition of X into $\mathcal{L}^{\dagger}_{\mathcal{A}}$-definable ordinary cells C_k, $k = 1, \ldots, q$, with centers given by $\mathcal{L}^{\dagger}_{\mathcal{A}}$-terms.
Furthermore, one can additionally require that each C_k is, after some coordinate permutation, a cell of type $(1, \ldots, 1, 0, \ldots, 0)$ with 1-Lipschitz centers $c_k = (c_{k,1}, \ldots, c_{k,n})$, $k = 1, \ldots, q$.

Some background

I prove that definable subsets of $\mathcal{O}_K^n$ admit, uniformly in definable families, certain Yomdin–Gromov T^r-parametrizations, and estimate their number.

The concept of Yomdin–Gromov parametrization goes back to their papers [Yo-1, Yo-2, Gro]. The crucial result, the Yomdin–Gromov algebraic lemma, provides also a polynomial estimate for the number of semialgebraic $\mathcal{C}^r$-parametrizations of a given bounded semialgebraic subset X of $\mathbb{R}^n$.

Its generalizations to the o-minimal settings (with remarkable applications in the fields of dynamics and Diophantine geometry), were initiated by Pila–Wilkie [PW], allowing them to establish their counting theorem about rational points on definable sets.

Some other approaches were recently given in the papers [CPV, BN1, BN2, CPW]. Note also that the notion of complexity is not available in the general o-minimal settings, and must be replaced by uniformity over families.

Basic terminology

For any valued field K, a subset P of $\mathcal{O}_K^m$ and a positive integer r, a function $f = (f_1, \ldots, f_n) : P \to \mathcal{O}_K^n$ is said to satisfy *T_r-approximation* if P is open in $\mathcal{O}_K^m$, and for each $a \in P$ there is an n-tuple $T_{f,a}^{<r}$ of polynomials with coefficients in $\mathcal{O}_K$ of degree $< r$ such that $\left| f(x) - T_{f,a}^{<r}(x) \right| < |x - a|^r$ for all $x \in P$.

Clearly, such polynomials $T_{f,a}^{<r}$ are unique, and if f is of class C^r, they are just the Taylor polynomials of f at a of order r.

Obviously, f satisfies T_1-approximation if and only if f is 1-Lipschitz.

T_r-parametrizations will be achieved from T_1-ones by precomposing with power functions, with Cauchy's estimates playing a key role. Therefore the concept of T_1-parametrization should be strengthened to allow the use of Cauchy's estimates for the involved terms defined on open polydiscs.

Strong T_1-parametrization

Hence the following two cases, affecting the definition of strong approximation below, are encountered:

(V1) the value group vK is densely ordered, i.e. it has no smallest positive element 1;
(V2) conversely, vK has a smallest positive element 1.

Let K be a model of the theory $T_{\mathcal{A}}$. The function f as above is said to satisfy strong C^1-approximation if P is an open cell with zero centers, f is given on P by Henselian terms which are 1-Lipschitz on P. Moreover, in case (V2), those terms are required to be 1-Lipschitz on the associated box B_{as} to each box B of P. Here B_{as} is the box over the algebraic closure K_{alg} of L defined by the same condition as B over K.

Theorem on strong T_1-parametrization

Theorem

Consider an $\mathcal{L}_{\mathcal{A}}$-definable subset X of $\mathcal{O}_K^k \times \mathcal{O}_K^n$, regarded as a definable family

$$X_w = \{x \in K^n : \ (w, x) \in X\}, \quad w \in W \subset K^m.$$

Suppose that the set X_w is of dimension m for each $w \in W$. Then there exist a finite set I and a definable family $f = (f_{w,i})_{(w,i)\in W\times I}$ of definable functions

$$f_{w,i} : P_{w,i} \to X_w \quad \text{with} \quad P_{w,i} \subset \mathcal{O}_K^m,$$

such that $(f_{w,i})_{i\in I}$ is a strong T_1-parametrization of the fiber X_w for each $w \in W$.

Theorem on T_r-parametrization

Theorem

Consider an $\mathcal{L}_{\mathcal{A}}$-definable subset $X \subset K^k \times \mathcal{O}_K^n$, regarded as a definable family

$$X_w = \{x \in K^n : \ (w, x) \in X\}, \quad w \in W \subset K^m,$$

and a positive integer r. Suppose that the set X_w is of dimension m for each $w \in W$ and that the congruence invariant $b_r = [r]RV(K)$ is finite. Then there exist a positive integer $s = s(X)$, a finite set I of cardinality $\leqslant s \cdot b_r^m$, and a definable family $f = (f_{w,i})_{(w,i)\in W\times I}$ of definable functions

$$f_{w,i} : P_{w,i} \to X_w \quad \text{with} \quad P_{w,i} \subset \mathcal{O}_K^m,$$

such that $(f_{w,i})_{i\in I}$ is a T_r-parametrization of X_w for each $w \in W$.

□

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