Resolution of Varieties and Foliations by Torus Actions

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Resolution of Singularities and Foliations

Core Literature:

- **[McQ19]** M. McQuillan, *Very fast, very functorial, and very easy resolution of singularities*, 2019.
- **[ATW19]** D. Abramovich, M. Temkin, J. Włodarczyk, *Functorial embedded resolution via weighted blowings up*, 2019.
- **[W22]** J. Włodarczyk, *Functorial resolution by torus actions*, 2022.
- **[ABTW25]** D. Abramovich, A. Belotto, M. Temkin, J. Włodarczyk, *Principalization on logarithmically foliated orbifolds*, 2025.

Weights vs. Smooth Centers

The Natural Geometry of Weights: For a singularity like $x^2 + y^3 + z^4$, we normalize the weights so that

$$w(x^2 + y^3 + z^4) = 1.$$

Thus, we naturally assign rational weights:

$$w(x) = \frac{1}{2}, \quad w(y) = \frac{1}{3}, \quad w(z) = \frac{1}{4}.$$

The Failure of Smooth Centers: Classical resolution relies on smooth centers, which implicitly assign a weight of 1 to all coordinates. This can strictly worsen a singularity.

Example: Blowing up the smooth origin for the hypersurface $y_0^2 - y_1 y_2 y_3 y_4$ yields the strict transform:

$$(y_0')^2 - y_1' y_2' y_3' (y_4')^2,$$

which possesses a more complicated singularity.

Tracking Weights via Rational Rees Algebras

Instead of classical ideals, we track fractional weights algebraically using **Rational Rees Algebras**.

Definition: A *rational Rees algebra* is a finitely generated $\mathcal{O}_X$-algebra:

$$R = \bigoplus_{a \in \Gamma} R_a t^a \subset \mathcal{O}_X[t^{1/w_R}],$$

where $\Gamma \subset \mathbb{Q}_{\geq 0}$ is a finitely generated additive subsemigroup, and $w_R \in \mathbb{Q}_{>0}$ is the grading denominator.

Maximal Admissible Centers: Given local parameters $x_1, \ldots, x_k$ and rational weights $a_i \in \mathbb{Q}_{>0}$, they naturally generate a *Rees center*:

$$\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\text{int}}$$

Definition of the Invariant via Rees Algebras

We use these Rees centers to define the fundamental resolution invariant.

The Invariant of an Ideal: Taking the maximum over all choices of x_i and a_i, set

$$\mathrm{inv}_p(\mathcal{I}) := \max\left\{(a_1,\ldots,a_k) \mid \mathcal{I}t \subset \mathcal{O}_X[x_1t^{1/a_1},\ldots,x_kt^{1/a_k}]^{\mathrm{int}}\right\},$$

where the rational weights are ordered lexicographically ($a_1 \leq \ldots \leq a_k$).

If $\mathrm{inv}_p(\mathcal{I}) = (a_1,\ldots,a_k)$ and the algebra

$$\mathcal{A} = \mathcal{O}_X[x_1t^{1/a_1},\ldots,x_kt^{1/a_k}]^{\mathrm{int}}$$

satisfies $\mathcal{A} \supset \mathcal{I}t$, then $\mathcal{A}$ is called a *maximal admissible center* for $\mathcal{I}$.

Example: For $\mathcal{I} = (x^2+y^3+w^4)$, we have

$$\mathcal{I}t \subset \mathcal{O}_X[xt^{1/2}, yt^{1/3}, wt^{1/4}]^{\mathrm{int}} \implies \mathrm{inv}_0(\mathcal{I}) = (2,3,4).$$

From Rational Weights to Cobordant Blow-ups

The cobordant space geometrically realizes the maximal admissible center

$$\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\text{int}}.$$

1. **Extended Algebra:** Adjoin $t^{-1/w}$, where $w = \text{lcm}(a_1, \ldots, a_k)$:

$$\mathcal{A}^{\text{ext}} := \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}].$$

2. **Rescaling:** Under $t \mapsto t^w$, the weights become

$$w_i = \frac{w}{a_i} \in \mathbb{Z}.$$

3. **Full Cobordant Blow-Up:**

$$B = \text{Spec}_X\left(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}]\right) \longrightarrow X.$$

Definition of Full Cobordant Blow-Up

The full cobordant blow-up

$$B = \mathrm{Spec}_X\left(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}]\right) \to X$$

has global coordinates:

- $s = t^{-1}$, shifting the grading by 1 (cf. the Serre twist $\mathcal{O}_X(1)$).
- $x_i' = x_i t^{w_i} = x_i / s^{w_i}, \quad 1 \leq i \leq k.$
- $x_j' = x_j, \quad j > k.$

Exceptional Divisor:

$$D = V(t^{-1}) = V(s) \subset B.$$

Vertex:

$$V = \mathrm{Vert}(B) := V(x_1 t^{w_1}, \ldots, x_k t^{w_k}) = V(x_1', \ldots, x_k').$$

It is analogous to the vertex of an affine cone.

True Cobordant Blow-Up:

$$\sigma_+ : B_+ = B \setminus V \longrightarrow X.$$

Weighted blow-ups via Cobordant blow-ups

The cobordant blow-ups is not a birational transformation. It introduces the action of torus T.
One can recover the standard definition of the *weighted blow-up* to be

$$B_+/T \to X,$$

where B_+/T is a *geometric quotient* (space of orbits) and *stack-theoretic weighted blow-up*

$$[B_+/T] \to X,$$

for the stack-theoretic quotient $[B_+/T]$.

Remark. Weighted stack-theoretic blow-ups were introduced in the context of resolution of foliations by Panazzolo and in the resolution of varieties independently by McQuillan-Marzo **[McQ19]** and by Abramovich-Temkin-Włodarczyk **[ATW19]**.

Cobordant vs. Orbifold Weighted Blow-ups

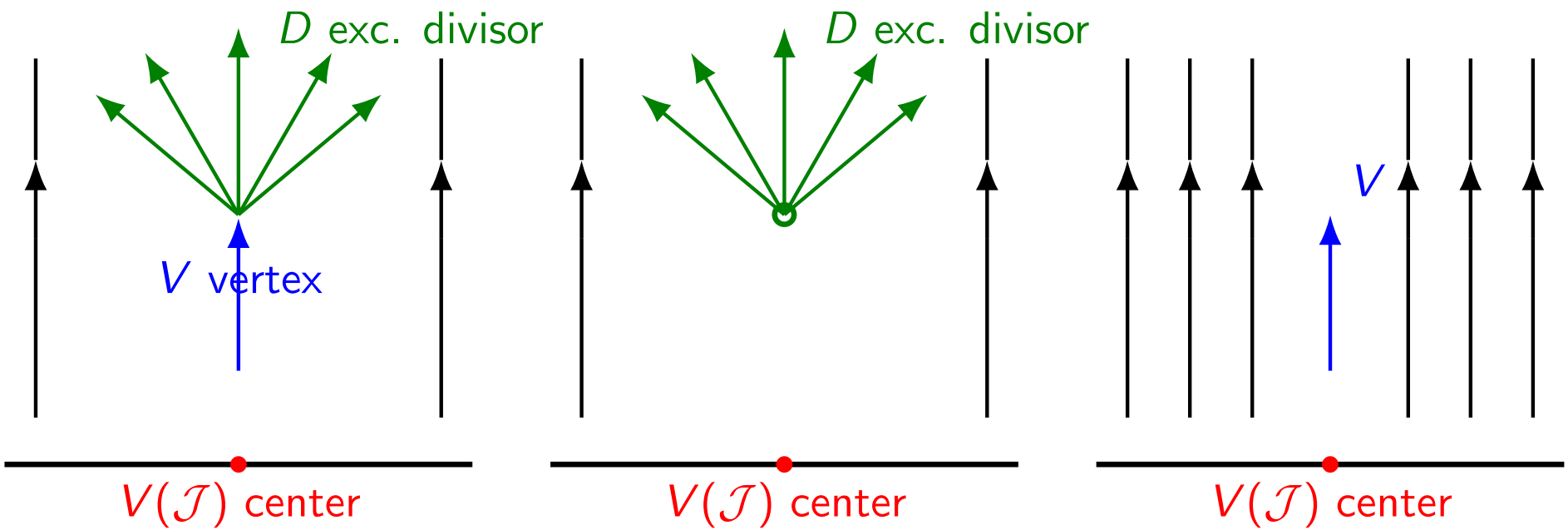
Cobordant blow-ups were introduced in [W22], and independently formulated as presentations of stacky weighted blow-ups by Quek and Rydh in [QR22].

Key Advantages Over Stacky Blow-ups:

- **Elimination of Stacks:** It replaces the abstract 2-categorical machinery of stacks with classical scheme theory equipped with a simple torus action.
- **Coordinate Simplicity:** It provides a global presentation requiring *only one chart*, bypassing the tedious gluing of multiple orbifold patches.
- **Foliation Compatibility:** It guarantees much better geometric properties when transforming and resolving derivations on foliated varieties.

Historical and Geometric Connections

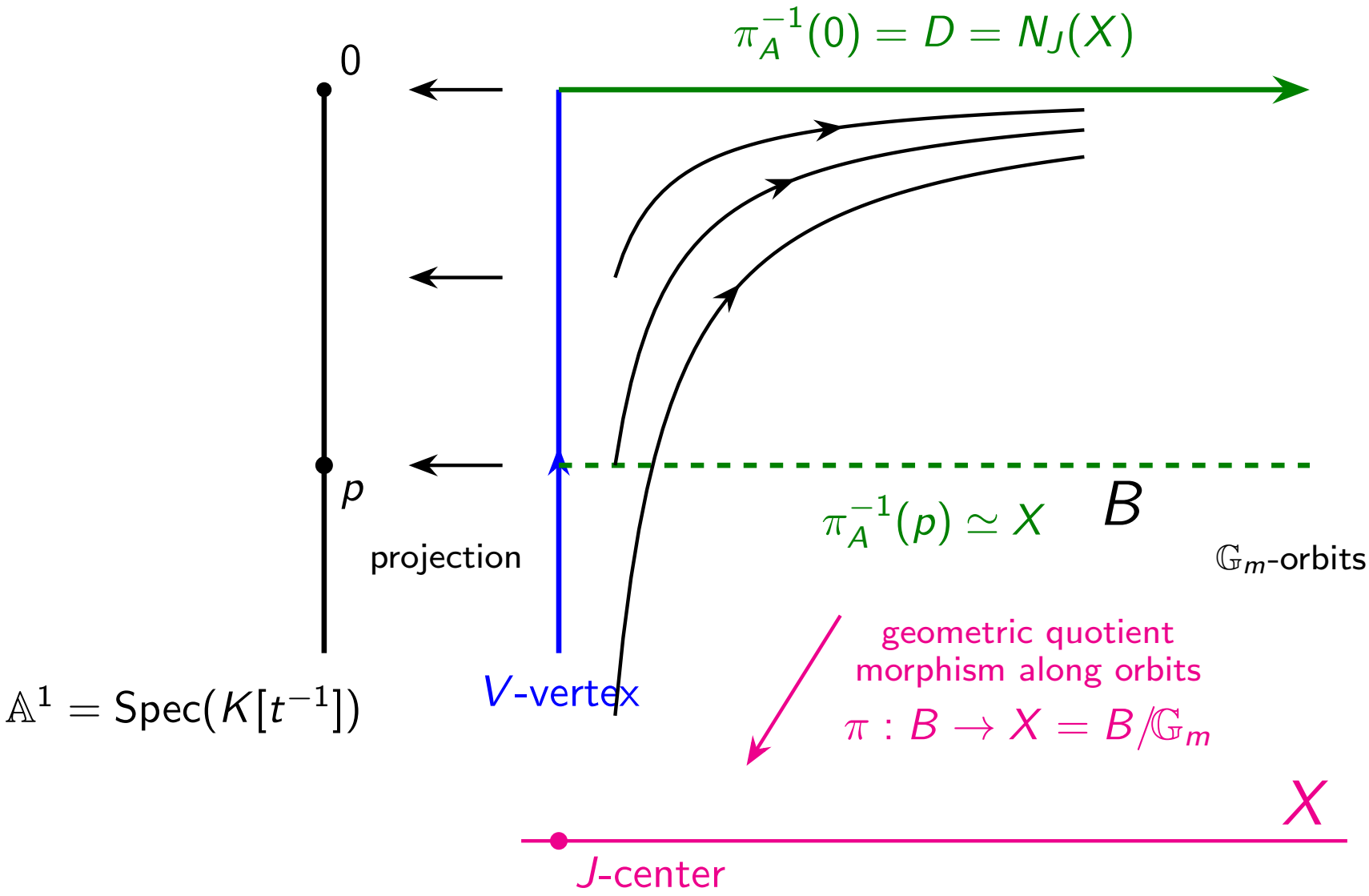
- **Algebraic Origin:** The definition originates from the concept of the **extended Rees algebra**, introduced by Rees.
- **Geometric Origin:** It naturally relates to the classical construction of the **deformation to the normal cone**, as developed and studied by MacPherson, Fulton, Reid, Kollár, and others.
- **Birational Cobordism:** The approach is fundamentally tied to the theory of **birational cobordism** (W), utilizing smooth spaces equipped with torus actions to represent and resolve complex birational morphisms.



B full cobord. blow-up $\quad B_+ = B \setminus V$ cob. blow-up $\quad B \setminus D = X \times \mathbb{G}_m$

Figure: Cobordant blow-up: the role of the vertex V and the exceptional divisor D.

Cobordant Blow-Up and Deformation to the Normal Bundle



The Resolution Principle

Rescaling the maximal admissibility inclusion:

$$\mathcal{I}t \subset \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]$$

gives

$$\mathcal{I}t^w \subset B = \mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}].$$

1. The Maximum Moves to the Vertex:
If $\mathrm{inv}_X(\mathcal{I})$ is maximal along the center, then the controlled transform

$$\sigma^c(\mathcal{I}) := \mathcal{O}_B \cdot \mathcal{I}t^{w_A}$$

attains the same maximum at the vertex V:

$$\max \mathrm{inv}_B(\sigma^c(\mathcal{I})) = \mathrm{inv}_B(\sigma^c(\mathcal{I}))|_V = \max \mathrm{inv}_X(\mathcal{I}).$$

2. Removing the Vertex Forces a Drop:
On $B_+ := B \setminus V$ the invariant **strictly drops**:

$$\max \mathrm{inv}_{B_+}(\sigma^c(\mathcal{I})) < \max \mathrm{inv}_X(\mathcal{I}).$$

Example: Brieskorn Singularities

Let $Y \subset X = \mathbb{A}^n_K$ be defined by the equation:

$$f = a_1 x_1^{b_1} + \ldots + a_n x_n^{b_n}$$

Assume $\mathrm{Sing}(Y) \subset V(x_1, \ldots, x_n)$. *(This is automatic in char. 0).*

1. The Rational Center & Rescaling

We define the maximal admissible center:

$$\mathcal{A} = \mathcal{O}_X[x_1 t^{1/b_1}, \ldots, x_n t^{1/b_n}]^{\mathrm{int}} \implies B = \mathrm{Spec}_X \left(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_n t^{w_n}] \right)$$

2. Lifting the Ideal

We lift $\mathcal{I}_Y$ to B using the invariant coordinates $x'_i = x_i t^{w_i}$. Factoring out the exceptional divisor t^{-w} yields:

$$\mathcal{O}_B \cdot \mathcal{I}_Y = \underbrace{t^{-w}}_{\text{exc. div}} \cdot \underbrace{\left(a_1 (x'_1)^{b_1} + \ldots + a_n (x'_n)^{b_n} \right)}_{\sigma^s(\mathcal{I}_Y) - \text{strict transform } Y'}$$

Resolution by Removing Singularities

The Geometric Payoff:
Notice that the strict transform Y':

$$\sigma^s(\mathcal{I}_Y) = \left(a_1 \left(x_1'\right)^{b_1} + \ldots + a_n \left(x_n'\right)^{b_n}\right)$$

has exactly the same equation as $\mathcal{I}_Y$.

Because the original singularities were confined to the origin, the singularities of the strict transform are confined entirely to the vertex of B:

$$\mathrm{Sing}(Y') \subset V(x_1', \ldots, x_n') = V$$

Conclusion:
The cobordant blow-up explicitly removes this vertex ($B_+ = B \setminus V$).
Therefore, the strict transform on B_+ is perfectly regular!

Resolution Process Defined by the Invariant

Process of Resolution:

- The maximum value of the invariant drops after each cobordant blow-up.
- The process continues until the invariant reaches the generic smooth point value:

$$\mathrm{inv}_p(\mathcal{I}_Y) = (1, \ldots, 1).$$

- At this stage, singularities on the strict transform of Y are completely resolved.

From Varieties to Foliations

The method for resolving foliated varieties and integrable foliations is a natural extension of desingularization using weighted centers.

Varieties as a Special Case:
The resolution of algebraic varieties via weighted centers can be reinterpreted as the resolution of a foliated variety $(X, \mathcal{F})$, where the foliation is the full tangent sheaf:

$$\mathcal{F} = \mathcal{D}_X.$$

A Unified Framework:
The same weighted centers, cobordant spaces, and admissibility criteria extend naturally from varieties to foliations.

The theory also admits a logarithmic version in the presence of an SNC divisor.

Distributions and Foliations

Let X be a smooth variety.

- A **distribution** is a coherent subsheaf

$$\Delta \subset \mathcal{D}_X.$$

- A **foliation** $\mathcal{F} \subset \mathcal{D}_X$ is an involutive distribution, i.e. it is closed under the Lie bracket:

$$X^1, X^2 \in \mathcal{F} \quad \Longrightarrow \quad [X^1, X^2] \in \mathcal{F}.$$

- The case

$$\mathcal{F} = \mathcal{D}_X$$

recovers the usual theory of resolution of varieties.

Singularities of Foliations

Like varieties, foliations possess singular and nonsingular points.

A foliation $\mathcal{F}$ of rank r is **nonsingular** at a point $\mathfrak{a} \in X$ if, in suitable formal coordinates $x_1, \ldots, x_n$ at $\mathfrak{a}$, it is generated by

$$\partial_{x_1}, \ldots, \partial_{x_r}.$$

Thus, formally near a nonsingular point, the leaves are simply

$$x_{r+1}, \ldots, x_n = \text{const}.$$

At a **singular point**, such a system of coordinates does not exist.

The Strategy: Resolution of Foliations

Let $(Y, \mathcal{G})$ be a foliated variety. The core strategy is to embed it rationally:

$$(Y, \mathcal{G}) \hookrightarrow (X, \mathcal{F})$$

such that:

- X is smooth.
- $\mathcal{F}$ is a *good foliation* (e.g., smooth, $\mathbb{K}$-monomial, or log-smooth).
- $\mathcal{F}|_Y = \mathcal{G}$.

By resolving the ideal $\mathcal{I}_Y$ on $(X, \widetilde{\mathcal{F}})$, we achieve:

- $\widetilde{\mathcal{F}}$ remains good.
- The strict transform $Y' \subset X'$ becomes $\widetilde{\mathcal{F}}$-aligned.
- The restricted foliation $\widetilde{\mathcal{F}}|_{Y'} = \mathcal{G}'$ is good (resolved).

Foliations and $\mathcal{F}$-Aligned Centers

To make the resolution compatible with the foliation, we require the centers to align with the foliation's structure.

Definition: Let $(X, \mathcal{F})$ be a foliated logarithmic variety. An adapted center $\mathcal{A}_J$ is $\mathcal{F}$-**aligned** if:

- **Formal Presentation:** There exist nested-regular coordinates (x, y) at p adapted to E such that:

$$\mathcal{A}_J \cdot \widehat{\mathcal{O}}_{X,(x)} = \widehat{\mathcal{O}}_{X,(x)}[x_i t^{1/a_i}, y_j t^{1/b_j}]^{\text{int}}$$

  where x_i are free variables.

- **Foliation Splitting:** The foliation locally splits as:

$$\widehat{\mathcal{F}}_{(x)} = \text{span}(\partial_{x_1}, \ldots, \partial_{x_l}, \nabla_1, \ldots, \nabla_m)$$

  where the derivations ∇_j are independent of (x_i, ∂_{x_i}), and the y-part of the Rees algebra is strictly $\widehat{\mathcal{F}}_p$-invariant:

$$\widehat{\mathcal{F}}_{(x)} \left(\widehat{\mathcal{O}}_{X,(x)}[y_j t^{1/b_j}]^{\text{int}} \right) \subset \widehat{\mathcal{O}}_{X,(x)}[y_j t^{1/b_j}]^{\text{int}}$$

The Canonical Invariant $\mathrm{inv}_{p,\mathcal{F}}$

Definition: Let $\mathcal{I}$ be an ideal (or R a Rees algebra) on $(X, \mathcal{F})$. The canonical invariant $\mathrm{inv}_{p,\mathcal{F}}(\mathcal{I})$ is the maximal weight sequence:

$$\max\left\{(a_1, \ldots, a_k, \infty + b_1, \ldots, \infty + b_r)\right\}$$

such that there is an $\mathcal{F}$-**aligned** presentation:

$$\mathcal{I}t \subset \widehat{\mathcal{O}}_{X,p}[x_i t^{1/a_i}, y_j t^{1/b_j}]^{\mathrm{int}}.$$

Here ∞ is a formal symbol making every $\infty + b$ larger than all transverse weights.

- **Transverse Part:** x_i are transverse directions, with weights a_i.
- **Invariant Part:** y_j are $\mathcal{F}$-invariant directions, with weights $\infty + b_j$.

Examples of the Invariant

Let's look at how the presence of a foliation changes the invariant.

Without foliation ($\mathcal{F} = \mathcal{D}_X$):

$$f = x^2 + y^3 + z^4, \qquad ft \subset \mathcal{O}_X[xt^{1/2}, yt^{1/3}, zt^{1/4}]^{\text{int}}$$

Since all directions are transverse, the invariant is purely numerical:

$$\text{inv}_p(f) = (2, 3, 4)$$

In the presence of a foliation: Assume $\mathcal{F} = \text{span}\{x\partial_x, \partial_y, \partial_z\}$.

- The variables y and z are strictly transverse to $\mathcal{F}$.
- The variable x is $\mathcal{F}$-invariant ($x\partial_x(x) = x$).

Thus, the optimal alignment is:

$$ft \subset \mathcal{O}_X[yt^{1/3}, zt^{1/4}, xt^{1/2}]$$

yielding the invariant:

$$\text{inv}_{p,\mathcal{F}}(f) = (\underbrace{3, 4}, \underbrace{\infty + 2})$$

Embedded Resolution of Foliated Varieties

- The maximum value of the invariant drops after each $\mathcal{F}$-aligned blow-up.
- The algorithm terminates when the invariant reaches the smooth point value:

$$\mathrm{inv}_p(\mathcal{I}_Y) = (1, \ldots, 1, \infty + 1, \ldots, \infty + 1)$$

- At this stage, the singularities on the strict transform of Y are resolved, and it is $\mathcal{F}$-aligned.
- Locally, in the coordinate system $x_1, \ldots, x_k, y_1, \ldots, y_{n-k}$, we have:

$$Y = V(x_1, \ldots, x_k, y_1, \ldots, y_m) \quad (m \leq n - k)$$

and

$$\widehat{\mathcal{F}} = \mathrm{span}(\partial_{x_1}, \ldots, \partial_{x_k}, \nabla_1(y), \ldots, \nabla_r(y))$$

where $V(x_1, \ldots, x_k)$ is strictly $\mathcal{F}$-transverse and $V(y_1, \ldots, y_m)$ is strictly $\mathcal{F}$-tangential.

$\mathcal{K}$-Monomial Foliations

Definition (Belotto)

Let $(X, \mathcal{F})$ be a foliated variety over $\mathbb{K}$, and let $\mathcal{K} \subset \mathbb{K}$ be a subfield. A foliation $\mathcal{F}$ is said to be *$\mathcal{K}$-monomial* at a point $\mathfrak{a}$ if there exists a regular coordinate system (w, v) in a neighborhood U of $\mathfrak{a}$ such that $\mathcal{F}$ can be locally generated on U by:

$$\partial_{v_i} \quad \text{and} \quad \nabla_j = \sum_{k=1}^{n} a_{jk} w_k \, \partial_{w_k},$$

where $j = 1, \ldots, n$, $i = 1, \ldots, r$, and $a_{jk} \in \mathcal{K}$.
Alternatively, in terms of regular forms, $\mathcal{F}$ can also be expressed as:

$$\prod_{j=1}^{n} w_j \cdot \sum_{j=1}^{n} \beta_{ij} \frac{dw_j}{w_j}(\partial) \equiv 0, \quad i = 1, \ldots, r,$$

where $\beta_{ij} \in \mathcal{K}$.

Globally $\mathcal{K}$-Darboux Totally Integrable Foliations

Definition

A foliation $\mathcal{F}$ on X is *totally integrable* if it is the foliation by fibers of a rational (or meromorphic) map

$$\varphi : X \dashrightarrow B,$$

i.e. the inverse transform of the zero foliation on B.

Definition

Let $(X, \mathcal{F})$ be a foliated variety over $\mathbb{K}$ and $\mathcal{K} \subset \mathbb{K}$ be a field. We say that $\mathcal{F}$ is *globally $\mathcal{K}$-Darboux totally integrable* if:

- there exists a rational (or meromorphic) morphism $\varphi : X \dashrightarrow B$
- there exists a $\mathcal{K}$-monomial foliation $\mathcal{G}$ on a smooth variety B;
- $\mathcal{F}$ is the inverse transform of $\mathcal{G}$ by φ.

Presentation of Darboux First Integrals

Classical Description:

A foliation $\mathcal{F}$ admits global $\mathcal{K}$-Darboux first integrals if it is defined by the vanishing of logarithmic forms:

$$\sum_{j=1}^{n_i} \beta_{ij} \frac{df_{ij}}{f_{ij}}(\partial) \equiv 0 \quad \text{for } i = 1, \ldots, r$$

where $f_{ij} \in K(X)$ and $\beta_{ij} \in \mathcal{K}^*$.

- **The Morphism:** The functions f_{ij} naturally define a rational map to a product of projective spaces:

$$\phi : X \dashrightarrow \prod_{i=0}^{r} \mathbb{P}^{n_i}$$

- **The Target Foliation:** The constants β_{ij} canonically determine a $\mathcal{K}$-monomial foliation $\mathcal{G}$ on the target space.
- **The Equivalence:** The original foliation is exactly the saturated pullback: $\mathcal{F} = \phi^{-1}(\mathcal{G})$.

Resolution of Rationally Totally and $\mathcal{K}$-Darboux Integrable Foliations

Theorem (ABTW)

There exists a resolution of a Totally (resp. $\mathcal{K}$-Darboux) Integrable Foliation $(X, \mathcal{F})$:

$$(X', \mathcal{F}') \to (X, \mathcal{F})$$

which is nonsingular in the cobordant case and $\mathcal{K}$-monomial in the weighted case, such that:

- *X' admits a torus action by T, with a geometric quotient X'/T and a T-invariant projective birational morphism $\pi : X'/T \to X$.*
- Cobordant Case: *The strict transform $\mathcal{F}' := \pi^s(\mathcal{F})$ is a* nonsingular *T-invariant foliation on the scheme X'.*
- Weighted Case: *The foliation descends to a* $\mathcal{K}$-monomial *foliation on the stack-theoretic quotient $[X'/T]$.*

The Mechanism: Resolution via the Graph

Let $\phi : X \dashrightarrow B$ be a dominant rational map, and $\mathcal{F} = \phi^{-1}(\mathcal{G})$ be the pull-back of the $\mathcal{K}$-monomial foliation $\mathcal{G}$ on a smooth B.

- Consider the product $Y = X \times B$ with projection $p_B : Y \to B$.
- Let $\overline{X} := \Gamma(\phi) \subset Y$ denote the graph of ϕ. The map $\overline{X} \to X$ is proper and birational.

Define $\mathcal{H}$ on $Y = X \times B$ to be the inverse transform $\mathcal{H} = p_B^{-1}(\mathcal{G})$. Then $\mathcal{H}$ is $\mathcal{K}$-monomial. In the **Cobordant case**, we resolve $\mathcal{H}$ to be nonsingular.

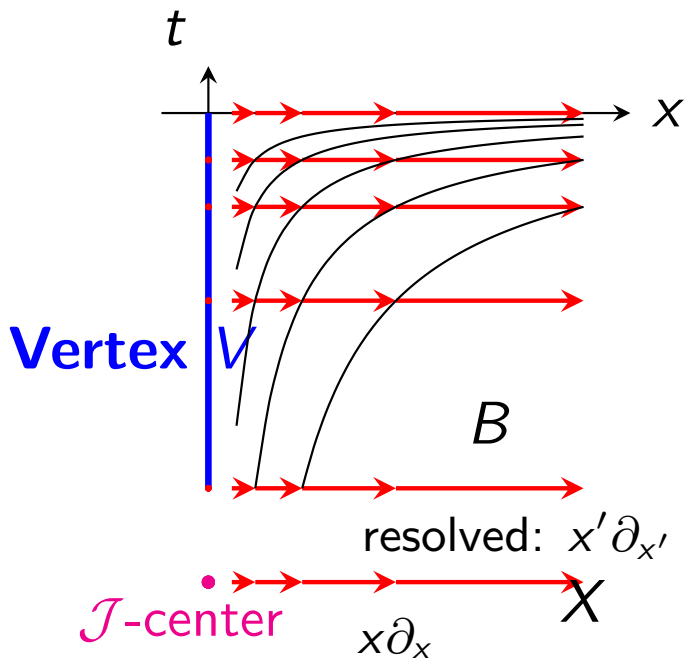
Applying Embedded Resolution of the Foliated Variety $(\overline{X} \subset Y, \mathcal{H})$**:**

- The strict transform $\widetilde{X}$ of $\overline{X}$ becomes $\mathcal{H}$-aligned.
- **Cobordant case:** The restriction $\overline{\mathcal{F}} = \mathcal{H}_{|\widetilde{X}}$ becomes nonsingular.
- **Weighted case:** We keep $\mathcal{H}$ as $\mathcal{K}$-monomial; the restriction $\overline{\mathcal{F}}$ remains $\mathcal{K}$-monomial.

Cobordant Resolution of a Singular Foliation

Example: The vector field $\mathcal{F} = x\partial_x$ is singular at the origin on $X = \mathbb{A}^1$.

- Consider the cobordant space
  $B = \mathrm{Spec}(\mathcal{O}_X[xt, t^{-1}]) = \mathrm{Spec}(\mathcal{O}_X[x', t^{-1}])$, with $x' = xt$.
- Removing the vertex $V = V(x')$ gives $B_+ = B \setminus V$.
- The strict transform becomes $\mathcal{F}' = x'\partial_{x'}$, which is nonsingular on B_+.



Thank you!

Thank you for your attention.