Recent developments in real algebraic geometry

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Preliminaries

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Theorem (Akbulut-King 1992)
Let $M \subset \mathbb{R}^m$ be a smooth compact connected manifold. Then, after a small deformation, M becomes a nonsingular connected component X_0 of a real affine variety $X \subset \mathbb{R}^m$.

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We don't know how to solve it in general even in the simplest cases $X = \mathbb{R}P^n$ or $X = \mathbb{S}^n$ for $n \geq 6$.

Algebraic approximation, complex case

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Complex approximation problem (Cornalba-Griffiths, 1975)
Let $X \subset \mathbb{C}^n$ be a smooth complex affine variety, let $U \subset X$ be a Runge open subset of X and let $M \subset U$ be a complex submanifold of U. Decide whether M admits algebraic approximation in X.

Codimension one

Let X be a smooth complex affine variety. Denote by $H^2_{\mathrm{alg}}(X;\mathbb{Z})$ the subgroup of $H^2(X;\mathbb{Z})$ generated by classes Poincaré dual to fundamental classes of codimension one subvarieties of X (i.e. the image of the cycle class map $\mathsf{CI}(X) \to H^2(X;\mathbb{Z})$).

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Theorem (Cornalba-Griffiths, 1975)
Let X be a smooth complex affine variety, U be its Runge open subset and $M \subset U$ be a complex submanifold of U of codimension one. Then M admits algebraic approximation in X if and only if the cohomology class in $H^2(U;\mathbb{Z})$ represented by M is the restriction of a class from $H^2_{\mathrm{alg}}(X;\mathbb{Z})$.

Codimension one

Let X be a nonsingular compact real affine variety. Denote by $H^1_{\mathrm{alg}}(X;\mathbb{Z}/2)$ the subgroup of $H^1(X;\mathbb{Z}/2)$ generated by classes Poincaré dual to fundamental classes of codimension one subvarieties of X.

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Theorem (Benedetti-Tognoli 1982)
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Example
If $X = \mathbb{R}P^n$, or $X = \mathbb{S}^n$ for some $n \geq 0$, then every codimension one submanifold of X admits algebraic approximation in X. Indeed, in both cases $H^1_{\mathrm{alg}}(X;\mathbb{Z}/2) = H^1(X;\mathbb{Z}/2)$.

Codimension two

Theorem (B., 2026)
Let X be a smooth complex affine variety, U be its Runge open subset and $M \subset U$ be a complex submanifold of U of codimension two. Then M admits algebraic approximation in X if and only if the K-theory class in $KU^0(U)$ corresponding to M is the restriction of a class from $KU^0_{\mathrm{alg}}(X)$.

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Codimension two examples

Example (Benoist, 2024)
Let

$$i : \mathbb{R}P^2 \hookrightarrow \mathbb{R}^4 \hookrightarrow \mathbb{R}P^1 \times \mathbb{R}P^3$$

be an embedding of $\mathbb{R}P^2$ (as a smooth manifold) into an affine chart of $\mathbb{R}P^1 \times \mathbb{R}P^3$. Then $i(\mathbb{R}P^2)$ does not admit algebraic approximation in $\mathbb{R}P^1 \times \mathbb{R}P^3$.

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Example continued (B., 2026)
The Klein bottle can be embedded into $\mathbb{R}^4$ in such a way that it admits algebraic approximation in $\mathbb{R}P^1 \times \mathbb{R}P^3$. However, it can also be embedded into $\mathbb{R}^4$ in such a way that it *does not* admit algebraic approximation in $\mathbb{R}P^1 \times \mathbb{R}P^3$.

The case of large codimension

Theorem (B., 2026)
Let X be a smooth complex affine variety, U be its Runge open subset and $M \subset U$ be a complex submanifold of U of dimension d. Assume that $2d < \dim X$. Then M admits algebraic approximation in X.

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Theorem (Benoist, 2024)
Let X be a nonsingular real affine variety and $M \subset X$ be a smooth submanifold of X of dimension d. Assume that $2d < \dim X$. Then M admits algebraic approximation in X if and only if its class in $MO_d(X)$ lies in $MO_d^{\mathrm{alg}}(X)$.

Conjectures

Conjecture I
Let X be a smooth complex affine variety, U be its Runge open subset and $M \subset U$ be a complex submanifold of U of codimension c.

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$$\Omega^c(X) \to MU^{2c}(X)$$

from algebraic cobordism.

Conjectures

Conjecture II

Let X be a nonsingular compact real affine variety and $M \subset X$ be a smooth submanifold of X of codimension c. Then M admits algebraic approximation in X if and only if the Real cobordism class in $M\mathbb{R}^{c,c}(X)$ corresponding to M is lies in the image of the natural comparison map

$$\Omega^c(X^{\mathbb{C}}) \to M\mathbb{R}^{c,c}(X^{\mathbb{C}}) \to M\mathbb{R}^{c,c}(X).$$

Here $X^{\mathbb{C}}$ is the complexification of X, i.e. its Zariski closure in $\mathbb{C}^n$ considered as a scheme over $\mathbb{R}$.

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The “only if” part in both conjectures follows simply from the definitions.

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The "only if" part in both conjectures follows simply from the definitions. Both of the conjectures are true if $c = 1, 2$ and if $2c > \dim X$.

Thank you for your attention!

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