ŁOJASIEWICZ 100

SEPTEMBER 21-25, 2026

KRAKÓW

APPROXIMATION OF MAPS BETWEEN REAL ALGEBRAIC VARIETIES

JULIUSZ BANECKI AND WOJCIECH KUCHARZ

TERMINOLOGY

By a real algebraic variety, we mean a quasiprojective real variety. Morphisms of real algebraic varieties are called regular maps.

Each real algebraic variety in the sense used here is actually affine, that is, isomorphic to an algebraic set in some $\mathbb{R}^n$. This fact allows us to describe regular functions and regular maps in a simple way.

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^p$ be algebraic sets, and let $U \subseteq X$ be a Zariski open set.

A function

$$\varphi : U \longrightarrow \mathbb{R}$$

is regular iff there exist two polynomial functions $P, Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$Q(x) \neq 0 \quad \text{and} \quad \varphi(x) = \frac{P(x)}{Q(x)} \quad \text{for all } x \in U.$$

A map

$$f = (f_1, \ldots, f_p) : U \longrightarrow Y \subseteq \mathbb{R}^p$$

is regular iff its components $f_1, \ldots, f_p$ are regular functions.

CONVENTION

Each real algebraic variety is endowed with the Zariski topology and the Euclidean topology. Unless explicitly stated otherwise, we use the Euclidean topology.

DEFINITION

Let X and Y be nonsingular real algebraic varieties, and let $k = 0, 1, \ldots, \infty$. We say that a $\mathcal{C}^k$ map $f: X \to Y$ can be approximated in the $\mathcal{C}^k$ topology by regular maps if, for every nbhd $\mathcal{U} \subseteq \mathcal{C}^k(X,Y)$ of f, there exists a regular map $g: X \to Y$ in $\mathcal{U}$.

The $\mathcal{C}^k$ topology referred to here is also known as the compact-open $\mathcal{C}^k$ topology or the weak $\mathcal{C}^k$ topology.

THE WEIERSTRASS APPROXIMATION THEOREM

Every $\mathcal{C}^k$ map $X \to \mathbb{R}^p$ can be approximated in the $\mathcal{C}^k$ topology by regular maps.

GOAL

Extend the Weierstrass theorem to Y-valued maps.

This cannot be done directly since we encounter the following phenomena:

- If $f \in \mathcal{C}^0(X,Y)$ can be approximated in the $\mathcal{C}^0$ topology, then f is homotopic to a regular map.
- If Y is compact connected with $\dim Y \geqslant 1$, then there exists a compact nonsingular real algebraic variety X and a $\mathcal{C}^\infty$ map $f: X \longrightarrow Y$ such that f is not homotopic to any regular map (J. Bochnak, WK, 2007).
- If $X = \mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$ and $Y = \{(x,y) \in \mathbb{R}^2 : x^4+y^4 = 1\}$, then every regular map $X \longrightarrow Y$ is constant. In particular, among $\mathcal{C}^\infty$ maps $X \longrightarrow Y$, only those that are constant can be approximated by regular maps. Note that if $X \longrightarrow Y$ is not surjective, it is homotopic to a regular map.

DEFINITION (APPROXIMATION PROPERTY)

Let Y be a nonsingular real algebraic variety. We say that Y has the approximation property if,

(i) for every nonsingular real algebraic variety X, and

(ii) every $\mathcal{C}^\infty$ map $f: X \to Y$ such that f is homotopic to a regular map $X \to Y$,

the map f can be approximated in the $\mathcal{C}^\infty$ topology by regular maps $g: X \to Y$.

QUESTION

What are examples of real algebraic varieties with the approximation property?

I will describe two classes of such varieties.

DEFINITION (HOMOGENEOUS SPACE)

A real algebraic variety Y is a homogeneous space for a linear algebraic group $G \subseteq GL(m,\mathbb{R})$ if it admits a transitive algebraic action

$$G \times Y \longrightarrow Y, \quad (a,y) \longmapsto a \cdot y.$$

Every homogeneous space is nonsingular.

When we refer to homogeneous spaces, we mean homogeneous spaces for linear algebraic groups.

(1) The unit n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is a homogeneous space for $SO(n+1)$.

(2) $\mathbb{P}^n(\mathbb{R})$ is a homogeneous space for $GL(n+1,\mathbb{R})$.

(3) The Grassmann variety $Gr_k(\mathbb{R}^n)$ is a homogeneous space for $GL(n,\mathbb{R})$.

THEOREM (J. Bochnak, WK, 2022)

Every homogeneous space has the approximation property.

DEFINITION (RATIONAL AND UNIFORML RATIONAL VARIETY)

Let Y be an irreducible n-dimensional real algebraic variety. We say that Y is rational if it contains a nonempty Zariski open subset that is biregularly isomorphic to a Zariski open subset of $\mathbb{R}^n$. We say that Y is uniformly rational if every point in Y has a Zariski open nbhd that is biregularly isomorphic to a Zariski open subset of $\mathbb{R}^n$.

OPEN QUESTION (Gromov, in the complex setting, 1989)

Suppose that Y is rational and nonsingular. Is Y uniformly rational?

(1) $\mathbb{S}^n$, $\mathbb{P}^n(\mathbb{R})$, $Gr_k(\mathbb{R}^n)$ are uniformly rational.

(2) If Y is uniformly rational, then every blowup of Y along a nonsingular Zariski closed center is also uniformly rational.

THEOREM (J. Banecki, WK, 2025)

Every uniformly rational real algebraic variety has the approximation property.

REMARK

(1) It is not know whether every algebraic group $G \subseteq GL(m, \mathbb{R})$ is rational.

(2) There are real homogeneous spaces that are not rational. An appropriate example, due to, J.-L. Colliot-Thélène and J.-J. Sansuc, is sophisticated.

(3) There are uniformly rational real algebraic varieties that are not homogeneous spaces. A trivial example is $\mathbb{R}^2 \setminus \mathbb{S}^1$. A more interesting example is a blowup of $\mathbb{P}^2(\mathbb{R})$ at 2 points.

One of our main theorems is a characterization of real algebraic varieties with the approximation property. We begin by introducing the necessary notions.

DEFINITION

Let Y be a nonsingular real algebraic variety.

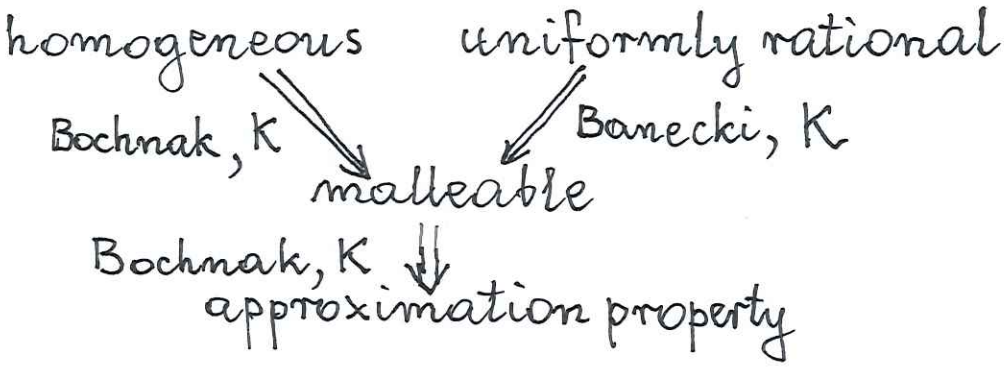
(1) (Spray). A spray (E,p,s) for Y consists of an algebraic vector bundle $p: E \to Y$ over Y and a regular map $s: E \to Y$ such that $s(0_y) = y$ for all $y \in Y$.

(2) (Dominating spray). A spray (E,p,s) for Y is said to be dominating if, for every $y \in Y$, the regular map

$$E_y \longrightarrow Y, \quad v \longmapsto s(v)$$

is a submersion at $0_y \in E_y$.

(3) (Malleable variety). The variety Y is called malleable if it admits a dominating spray.



approximation property $\overset{?}{\Longrightarrow}$ malleable

For any topological space T, let $\Omega(T)$ denote the collection of all relatively compact open subsets of T.

(4) (Quasi-spray). A quasi-spray $(E, p, \{s_V\}, \{\tau_V\})$ for Y consists of an algebraic vector bundle $p: E \to Y$ over Y and two collections $\{s_V\}, \{\tau_V\}$ indexed by $V \in \Omega(Y)$, where $s_V: E \to Y$ is a regular map and $\tau_V: V \to E$ is a $\mathcal{C}^\infty$ map such that

$$s_V(\tau_V(y)) = y \text{ and } p(\tau_V(y)) = y \text{ for all } y \in V.$$

(5) (Dominating quasi-spray). A quasi-spray $(E, p, \{s_V\}, \{\tau_V\})$ for Y is said to be dominating if, for every $V \in \Omega(Y)$ and every $y \in V$, the regular map

$$E_y \longrightarrow Y, \quad v \mapsto s_V(v + \tau_V(y))$$

is a submersion at $0_y \in E_y$.

(6) (Quasi-malleable variety). The variety Y is called quasi-malleable if it admits a dominating quasi-spray.

Note that

$$\text{malleable} \Longrightarrow \text{quasi-malleable}.$$

Indeed, if (E, p, s) is a dominating spray for Y, then $(E, p, \{s_V\}, \{\tau_V\})$, where $s_V = s$ and $\tau_V = 0$ for all $V \in \Omega(Y)$, is a dominating quasi-spray for Y.

THEOREM (J. Banecki, WK, 2025)

Let Y be a nonsingular real algebraic variety. Then the following conditions are equivalent:

(a) Y has the approximation property.

(b) Every $\mathcal{C}^\infty$ map $\sigma: TY \longrightarrow Y$, with $\sigma(0_y) = y$ for all $y \in Y$, can be approximated in the $\mathcal{C}^1$ topology by regular maps.

(c) Every $\mathcal{C}^\infty$ map $\varphi: Y \times \mathbb{R} \longrightarrow Y$, with $\varphi(y, 0) = y$ for all $y \in Y$, can be approximated in the $\mathcal{C}^1$ topology by regular maps.

(d) Y is quasi-malleable.

COROLLARY

Let Y be quasi-malleable, and let $n \in \mathbb{Z}_+$. Then every $\mathcal{C}^\infty$ map $\mathbb{R}^n \longrightarrow Y$ can be approximated in the $\mathcal{C}^\infty$ topology by regular maps.

Proof. It suffices to note that the last theorem is applicable since $\mathbb{R}^n \longrightarrow Y$ is null homotopic. □

We also have a result that combines approximation with interpolation.

DEFINITION (APPROXIMATION-INTERPOLATION PROPERTY)
Let Y be a nonsingular real algebraic variety. We say that Y has the approximation-interpolation property if,

(i) for every nonsingular real algebraic variety X,

(ii) every nonsingular Zariski closed subvariety $A \subseteq X$, and

(iii) every $\mathcal{C}^\infty$ map $f: X \longrightarrow Y$ such that $f|_A$ is a regular map and f is homotopic relative to A to a regular map $X \longrightarrow Y$,

the map f can be approximated in the $\mathcal{C}^\infty$ topology by regular maps $g: X \longrightarrow Y$ satisfying $g|_A = f|_A$.

THEOREM (J. Banecki, WK, 2025)
Let Y be a nonsingular real algebraic variety. Then the following conditions are equivalent:

(a) Y has the approximation-interpolation property.

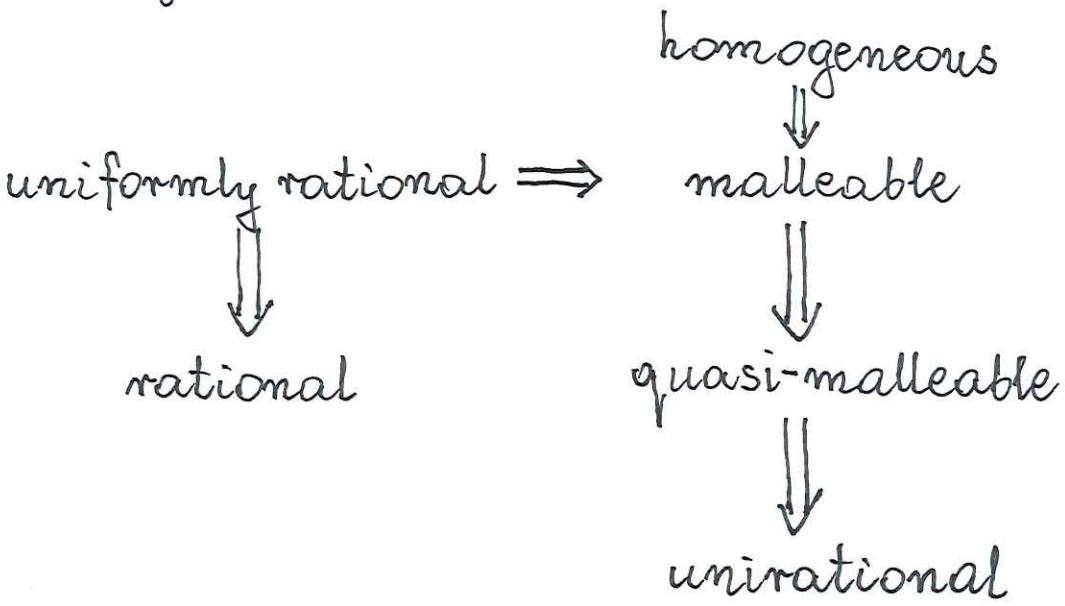
(b) Every $\mathcal{C}^\infty$ map $\sigma: TY \to Y$, with $\sigma(0_y) = y$ for all $y \in Y$, can be approximated in the $\mathcal{C}^1$ topology by regular maps $\tau: TY \to Y$ satisfying $\tau(0_y) = y$ for all $y \in Y$.

(c) Every $\mathcal{C}^\infty$ map $\varphi: Y \times \mathbb{R} \to Y$, with $\varphi(y,0) = y$ for all $y \in Y$, can be approximated in the $\mathcal{C}^1$ topology by regular maps $\psi: Y \times \mathbb{R} \to Y$ satisfying $\psi|_{Y \times \{0\}} = \varphi|_{Y \times \{0\}}$.

(d) Y is malleable.

As indicated above, malleability implies quasi-malleability. Because malleability is easier to verify, the approximation-interpolation property admits a simpler characterization than the approximation property. Whether quasi-malleability implies malleability remains an open question.

To conclude, let us note that the following implications hold for all irreducible nonsingular real algebraic varieties:



Recall that a real algebraic variet Y is called unirational if there exists a regular map $\varphi: U \longrightarrow Y$, defined on a Zariski open subset $U \subseteq \mathbb{R}^n$, for some n, whose image $\varphi(U)$ is Zariski dense in Y. In particular, a unirational variety is irreducible.