Singularities of Special Submanifolds in Symplectic Spaces

Goo Ishikawa (Hokkaido University, Japan)

A joint work with Stanisław Janeczko

Dedicated to the memory of
Professor Stanisław Łojasiewicz (1926–2002)

Łojasiewicz 100

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When I(Goo Ishikawa) was a graduate student in Kyoto University (Japan), I started to study, say, Catastrophe Theory, Singularity Theory of Differentiable Mappings, Real Algebraic Geometry, Differentiable Algebra and Differential Analysis, and tried to read, among others, the very famous paper

Ensembles Semi-Analytiques

by Professor Łojasiewicz:

E N S E M B L E S S E M I - A N A L Y T I Q U E S

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And, later, I have valuable opportunity to meet Prof. Łojasiewicz:



2001 Bedlewo

【 Special Submanifolds in a Symplectic Space 】

Let (M, ω) be a symplectic manifold of dimension $2n$ with symplectic form ω (closed, non-degenerate 2-form) e.g.,

$$\mathbb{R}^{2n} = \{(p, q)\}, \quad \omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

Special kinds of submanifolds (possibly with singularities) in a symplectic space:

isotropic, Lagrangian, coisotropic

$N \subset M$ submanifold, $TN^{\perp} := \{(y, v) \in TM \mid \omega(v, T_y N) = 0\}$, "symplectic orthogonal",

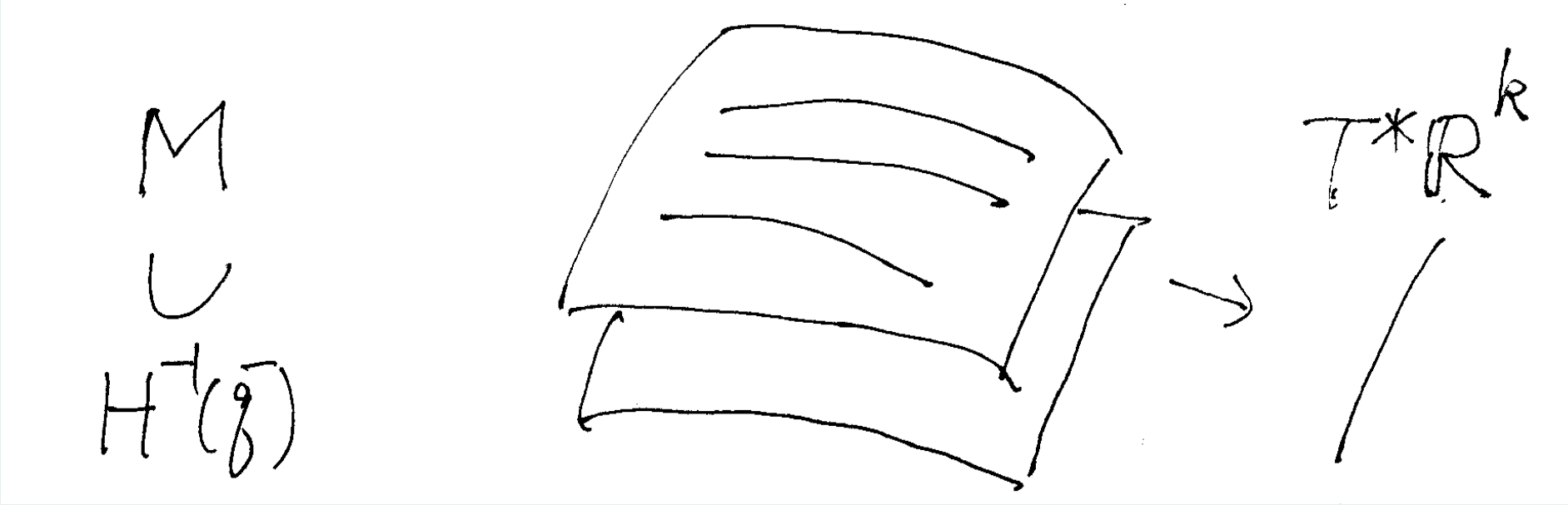
$$TN \subseteq TN^{\perp}, \quad TN = TN^{\perp}, \quad TN \supseteq TN^{\perp}.$$

Previous joint works of G. Ishikawa and S. Janeczko:

1. *Symplectic bifurcations of plane curves and isotropic liftings*, Quarterly Journal of Mathematics, Oxford (2003).
2. *Symplectic singularities of isotropic mappings*, Banach Center Publ. (2004).
3. *Bifurcations in symplectic space*, Banach Center Publ. (2008).
4. *Symplectic classification of parametric complex plane curve*, Annales Polonici Mathematici (2010).
5. *Symplectic invariants of parametric singularities*, Advances in Geometric Analysis, International Press (2012).
6. *Residual algebraic restrictions of differential forms*, Methods and Applications of Analysis (2017).
7. *Singular mappings and their zero-forms*, Pure and Appl. Math. Quarterly (2020).
8. *Symplectic singularities of differentiable mappings*, Surveys in Differential Geometry, International Press (2021).

【 Symplectic bifurcation problem 】

For a symplectic space (M, ω), consider a coisotropic fibration $H : M \to \mathbb{R}^{n-k}, 0 \leq k \leq n-1$, so that each fibre $H^{-1}(\bar{q})$, $\dim H^{-1}(\bar{q}) = n + k$, is a coisotropic submanifold for each $\bar{q} \in \mathbb{R}^{n-k}$. Then we have the fibre-wise symplectic reduction $\pi : M \to \mathbb{R}^{n-k} \times T^*\mathbb{R}^k$.



【 Symplectic bifurcation problem 】

The symplectic bifurcation problem is studied in the paper

Goo Ishikawa and Stanisław Janeczko,
Symplectic bifurcations of plane curves and isotropic liftings,
Quarterly Journal of Mathematics, Oxford (2003),

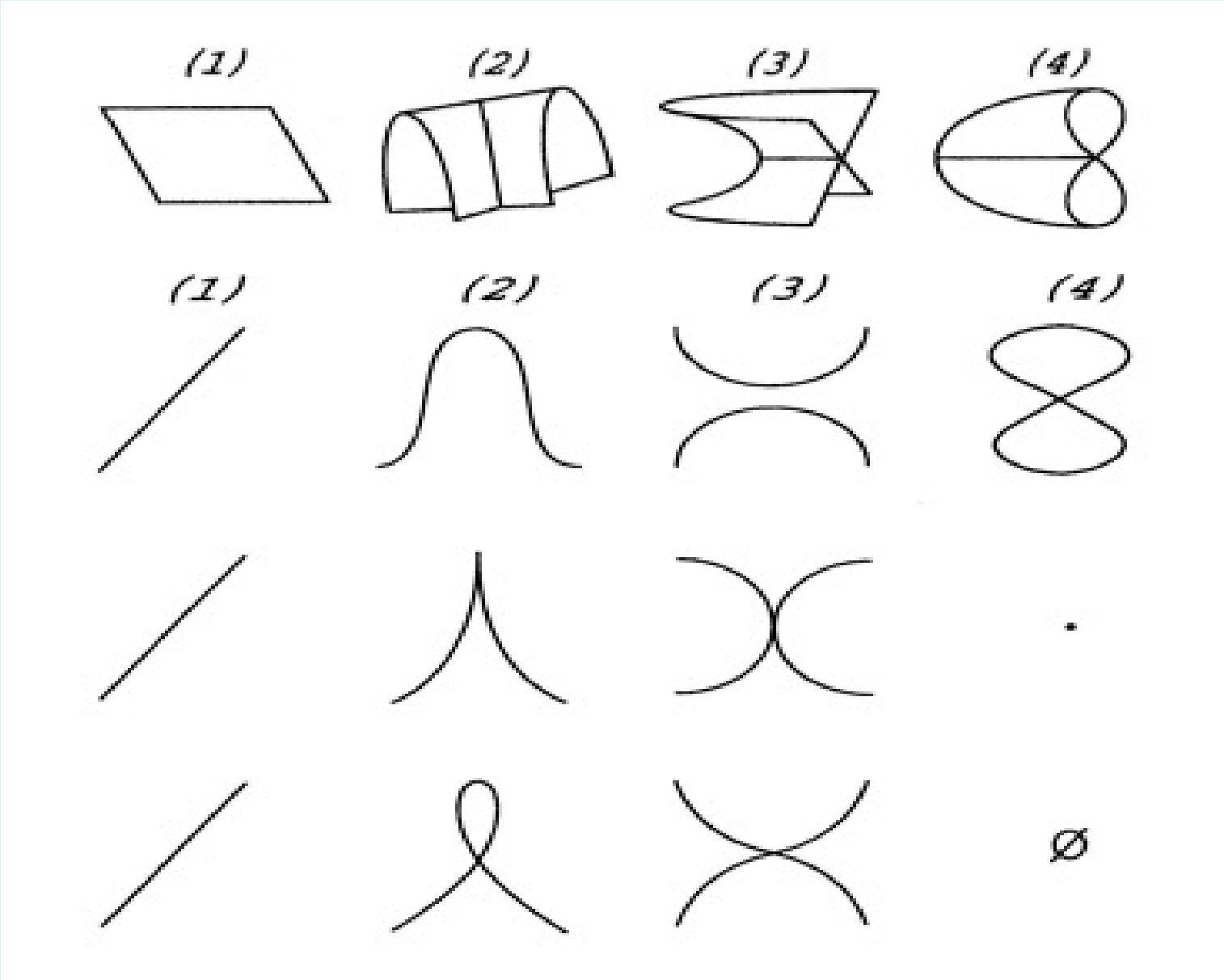
and in succeeding joint papers.

Let us recall some of results shortly.

Let $k = 1$. Then a symplectic bifurcation problem of plane curves is given by a map-germ $F : (\mathbb{R}^n, 0) \to \mathbb{R}^{n-1} \times T^*\mathbb{R}$.

The map-germ F is called transverse if F is transverse to (const.) $\times T^*\mathbb{R}$ at 0.

For instance, in the case $k = 1, n = 2$, we have generically:



【 Classification of symplectic plane curves 】

Let $\mathbb{R}^2 = T^*\mathbb{R} = \{(p,q)\}$ be a symplectic plane, with a non-degenerate (closed) differential 2-form ω, say $\omega = dp \wedge dq$.

Definition. Two map-germs $f, g : (\mathbb{R},0) \to \mathbb{R}^2$ are called symplectomorphic (resp. diffeomorphic) if there is a symplectomorphism-germ (resp. diffeomorphism-germ) $\tau : (\mathbb{R}^2, f(0)) \to (\mathbb{R}^2, g(0))$, $\tau^*\omega = \omega$, and a diffeomorphism-germ $\sigma : (\mathbb{R},0) \to (\mathbb{R},0)$ such that the diagram

$$\begin{array}{ccc} (\mathbb{R},0) & \xrightarrow{f} & (\mathbb{R}^2, f(0)) \\ \sigma \downarrow & & \downarrow \tau \\ (\mathbb{R},0) & \xrightarrow{g} & (\mathbb{R}^2, g(0)) \end{array}$$

commutes.

【 Codimensions of symplectic plane curves 】

Given a symplectic plane curve $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$, we introduce the symplectic codimension and the differential codimension by infinitesimal way:

Definition.

$$\begin{aligned} \text{sp-codim}(f) &:= \dim_{\mathbb{R}} V_f/(tf(V_1) + wf(VH_2)), \\ \text{diff-codim}(f) &:= \dim_{\mathbb{R}} V_f/(tf(V_1) + wf(V_2)). \end{aligned}$$

Here, in terms of Thom-Mather-Wall's notations, we denote by V_f the space of vector fields $(\mathbb{R}, 0) \to T\mathbb{R}^2$ along f, V_1 (resp. V_2) that of vector fields over $(\mathbb{R}, 0)$ (resp. $(\mathbb{R}^2, 0)$), $VH_2 \subset V_2$ that of Hamiltonian vector fields over the symplectic plane $(\mathbb{R}^2, 0)$.

Moreover, $tf : V_1 \to V_f$ is defined by $tf(\xi) = f_*(\xi)$ and $wf : V_2 \to V_f$ by $wf(\eta) = \eta \circ f$.

【 Algebraic invariants of parametric plane curves 】

For a symplectic plane curve $f = (p, q) : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$, we set

$$\begin{aligned} \mathcal{R}_f &:= \{h \in \mathcal{E}_1 \mid dh \in \langle dp, dq \rangle \mathcal{E}_1\}, \\ \mathcal{G}_f &:= \{h \in \mathcal{E}_1 \mid dh \in \langle dp, dq \rangle_{f^*\mathcal{E}_2}\}, \end{aligned}$$

where $f^* : \mathcal{E}_2 \to \mathcal{E}_1$ is the induced homomorphism between algebras of C^∞ function-germs by f. We have

$$\mathcal{E}_1 \supset \mathcal{R}_f \supset \mathcal{G}_f \supset f^*\mathcal{E}_2.$$

Then we have

Theorem.

$$\begin{aligned} \text{sp-codim}(f) &= \dim_{\mathbb{R}} \mathcal{E}_1/f^*\mathcal{E}_2 \ (= \delta(f)), \\ \text{diff-codim}(f) &= \dim_{\mathbb{R}} \mathcal{E}_1/\mathcal{G}_f, \end{aligned}$$

【 Symplectic ghost number (symplectic defect) 】

Definition. We call the deference

$$\mathrm{sd}(f) := \text{sp-codim}(f) - \text{diff-codim}(f),$$

the symplectic ghost number or the symplectic defect of f.

Theorem.

$$\begin{aligned} \text{sp-codim}(f) &= \dim_{\mathbb{R}} \mathcal{E}_1/f^*\mathcal{E}_2, \\ \text{diff-codim}(f) &= \dim_{\mathbb{R}} \mathcal{E}_1/\mathcal{G}_f, \\ \mathrm{sd}(f) &= \dim_{\mathbb{R}} \mathcal{G}_f/f^*\mathcal{E}_2 \end{aligned}$$

are diffeomorphism invariants of symplectic plane curves : If $g : (\mathbb{R},0) \to \mathbb{R}^2 = T^*\mathbb{R}$ is diffeomorphic to to f, then $\text{sp-codim}(g) = \text{sp-codim}(f)$, $\text{diff-codim}(g) = \text{diff-codim}(f)$ and thus $\mathrm{sd}(g) = \mathrm{sd}(f)$.

【 Symplectic classification of $\mathcal{A}$-simple plane curves 】

Theorem. Let $f : (\mathbb{R}, 0) \to \mathbb{R}^2$ be a smooth $(= C^\infty)$ map-germ to the symplectic plane $\mathbb{R}^2 = T^*\mathbb{R}$, which is $\mathcal{A}$-simple $(= 0$-modal) in the sense of Bruce-Gaffney, is symplectomorphic to one of the followings:

$$
\begin{array}{ll}
A_{2\ell}: & (t^2, t^{2\ell+1}), \\
E_{6\ell}: & (t^3, t^{3\ell+1} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1}), \\
E_{6\ell+2}: & (t^3, t^{3\ell+2} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1}), \\
W_{12}: & (t^4, t^5 + \lambda t^7), \\
W_{18}: & (t^4, t^7 + \lambda t^9 + \mu t^{13}), \\
W^{\#}_{1,2\ell-1}: & (t^4, t^6 + \lambda t^{2\ell+5} + \mu t^{2\ell+9}), \lambda \neq 0, (\ell = 1, 2, \dots).
\end{array}
$$

【 Symplectic versal deformations of plane curves 】

$A_{2\ell}$ (sp-codim $= \ell$) :

$(t^2,\ t^{2\ell+1} + \sum_{j=1}^{\ell} \lambda_j t^{2\ell-2j+1})$.

$E_{6\ell}$ (sp-codim $= 3\ell$) :

$(t^3+\lambda t,\ (\pm 1)^{\ell+1} t^{3\ell+1} + \sum_{j=1}^{\ell} \mu_j t^{3\ell-3j+1} + \sum_{j=1}^{2\ell-1} \nu_j t^{6\ell-3j-1})$.

$E_{6\ell+2}$ (sp-codim $= 3\ell+1$) :

$(t^3 + \lambda t,\ (\pm 1)^{\ell} t^{3\ell+2} + \sum_{j=1}^{\ell} \mu_j t^{3\ell-3j+2} + \sum_{j=1}^{2\ell} \nu_j t^{6\ell-3j+1})$.

W_{12} (sp-codim $= 6$) :

$(t^4 + \lambda_1 t^2 + \lambda_2 t,\ t^5 + \mu_1 t^7 + \mu_2 t^3 + \mu_3 t^2 + \mu_4 t)$.

W_{18} (sp-codim $= 9$) :

$(t^4 + \lambda_1 t^2 + \lambda_2 t,\ t^7 + \mu_1 t^{13} + \mu_2 t^9 + \mu_3 t^6 + \mu_4 t^5 + \mu_6 t^2 + \mu_7 t)$.

$W^{\sharp}_{1,2q-1}$ (sp-codim $= q+7$) :

$(t^4 + \lambda t^2 + \rho t,$

$\pm t^6 + t^{2q+5} + \mu t^{2q+9} + \sum_{j=0}^{q+2} \nu_j t^{2q+5-2j} + \theta t^2 + \rho t^{2q+2})$, $(\nu_0 \neq -1)$.

【 Examples 】

Example. A_4: symplectic (differential) normal form (t^2, t^5),
sp-codim = diff-codim = 2, sd = 0,
sym. (diff.) versal deformation $(t^2, t^5 + \lambda_1 t^3 + \lambda_2 t)$.

Example. E_6: symplectic (differential) normal form (t^3, t^4),
sp-codim = diff-codim = 3, sd = 0,
sym. (diff.) versal deformation $(t^3 + \lambda t, t^4 + \mu_1 t + \nu_1 t^2)$.

Example. W_{12}: symplectic normal form $(t^4, t^5 + \lambda t^7)$,
differential normal form $(t^4, t^5 \pm t^7), (t^4, t^5)$,
sp-codim $= 6, 6$, diff-codim $= 5, 6$, sd $= 1, 0$.
symplectic versal deformation:
$(t^4 + \lambda_1 t^2 + \lambda_2 t,\ t^5 + \mu_1 t^7 + \mu_2 t^3 + \mu_3 t^2 + \mu_4 t)$.

Lagrangian frontal

(a new subject of joint-works)

【 Symplectic plane frontals 】

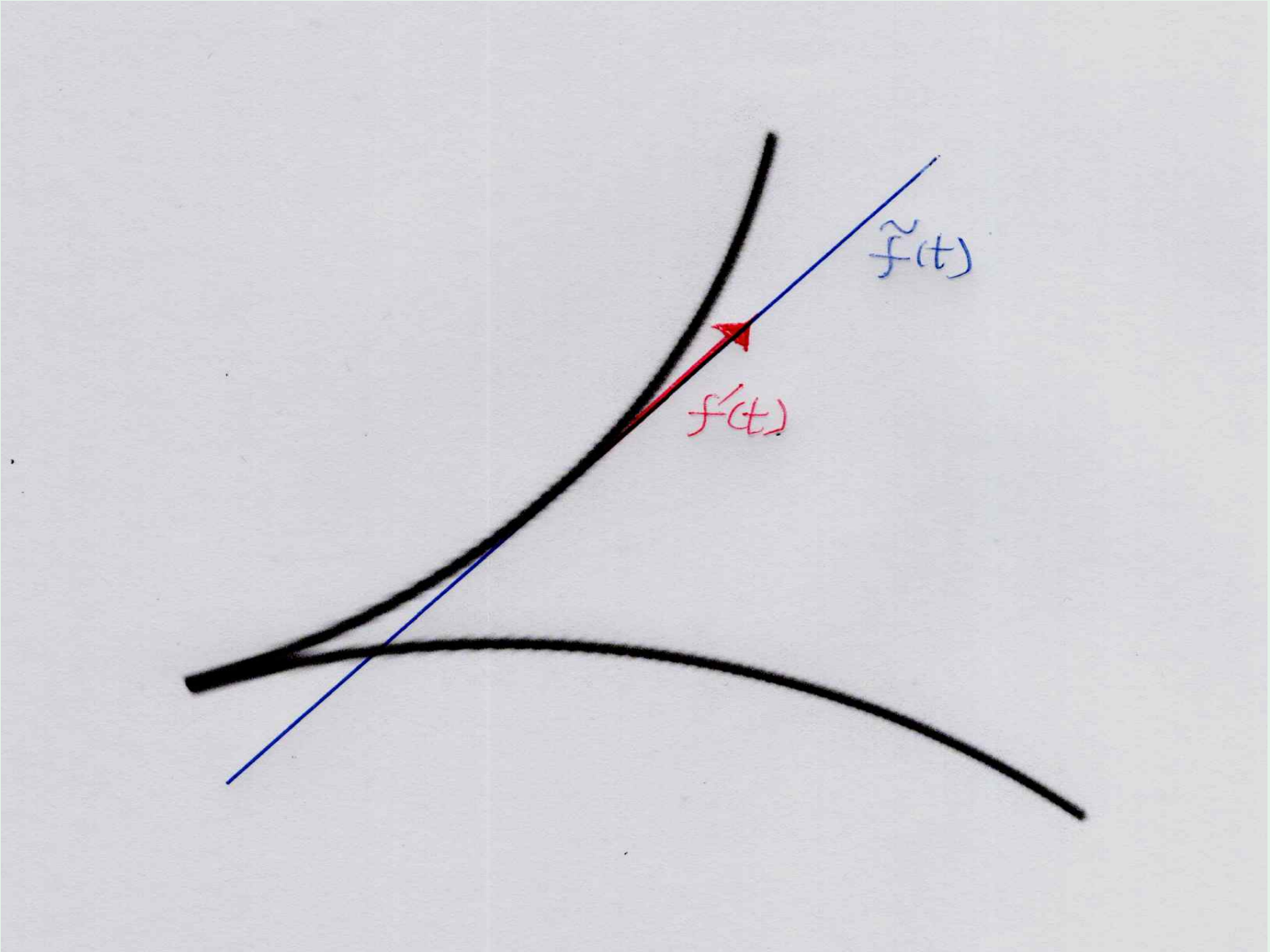
Now we would like to introduce the following notion:

Definition. The differentiable ($= C^\infty$) map-germ $f : (\mathbb{R}, 0) \to \mathbb{R}^2$ is called a frontal if there exists a smooth field $\widetilde{f}(t)$ of tangent lines along $f(t)$ such that

$$f'(t) \in \widetilde{f}(t) \subset T_{f(t)}\mathbb{R}^2.$$

A map-germ $f : (\mathbb{R}, 0) \to \mathbb{R}^2$ to the symplectic plane $(\mathbb{R}^2, \omega)$ is always isotropic, i.e., $f^*\omega = 0$. We call an isotropic frontal also a Lagrangian frontal.

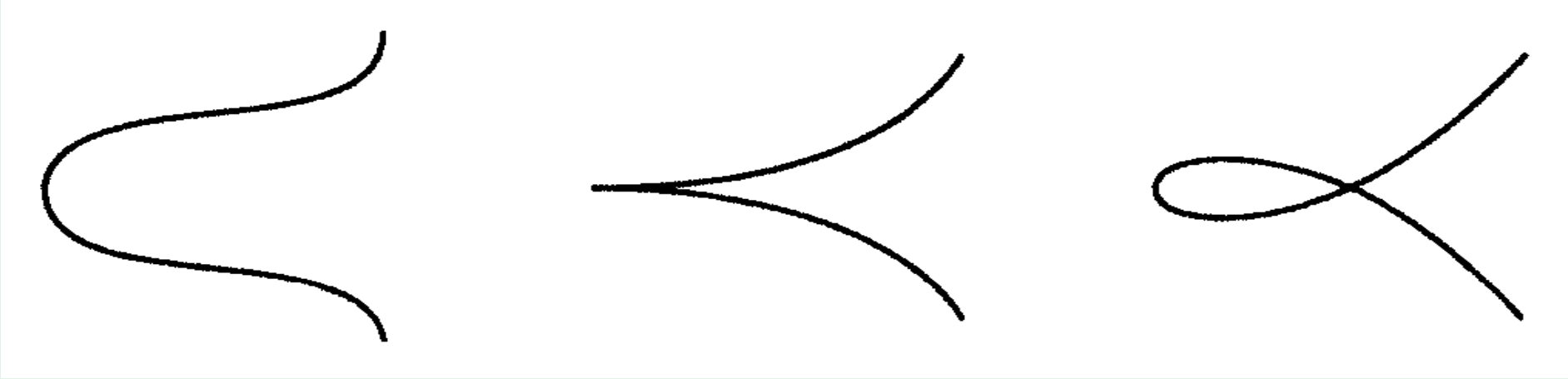
A frontal image on the symplectic plane :



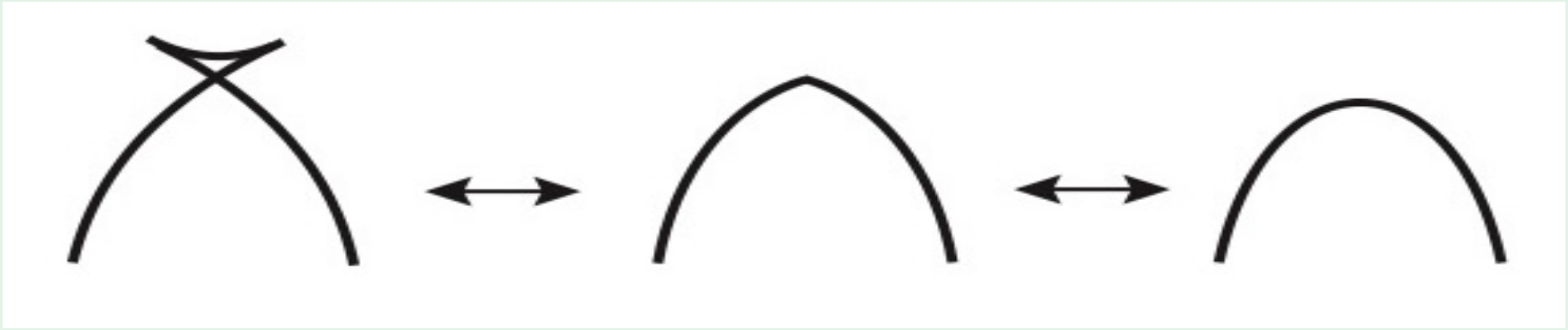
Almost all parametric C^∞ curves $(\mathbb{R}, 0) \to \mathbb{R}^2$ are frontals. In particular, any real analytic curve $(\mathbb{R}, 0) \to \mathbb{R}^2$ is a frontal.

However, under C^∞ deformations, tangent spaces need not behave in C^∞ nor even C^0 way:

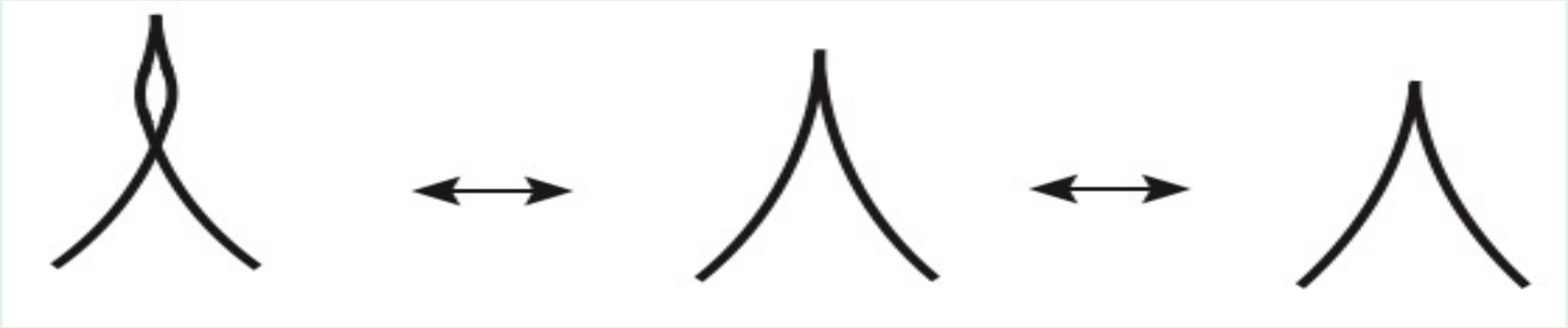
A non-frontal deformation of the cuspidal edge



A frontal deformations of the $(3,4)$-cusp (E_6)



A frontal deformations of the $(2,5)$-cusp (A_4)



【 Frontals and frontal deformations 】

Definition. A C^∞ map-germ $f : (\mathbb{R}^n, 0) \to \mathbb{R}^m$ $(n \leq m)$ is called a frontal if there exists a C^∞ family of tangential n-planes $\widetilde{f}(x) \subset T_{f(x)}\mathbb{R}^m$ such that $df(T_x\mathbb{R}^n) \subset \widetilde{f}(x)$ for $x \in (\mathbb{R}^n, 0)$.

Definition. A deformation $F : (\mathbb{R}^\ell \times \mathbb{R}^n, 0) \to \mathbb{R}^m$ of f, $F(u, x) = f_u(x), f_0 = f$, is called a frontal deformation of f if if there exists a C^∞ family of tangential n-planes $\widetilde{f_u}(x) \subset T_{f_u(x)}\mathbb{R}^m$ which depends smoothly on both x and u and satisfies $df_u(T_x\mathbb{R}^n) \subset \widetilde{f_u}(x)$ for $x \in (\mathbb{R}^n, 0)$.

【 A motivation — Frontal deformation theory 】

Problem: Construct the singularity theory on the frontal deformations of symplectic plane frontals.

【 Invariants of parametric plane frontals 】

For a symplectic plane curve $f = (p, q) : (\mathbb{R}, 0) \to (\mathbb{R}^2, \omega)$, we set

$$\begin{aligned} \mathcal{R}_f &:= \{h \in \mathcal{E}_1 \mid dh \in \langle dp, dq \rangle \mathcal{E}_1\}, \\ \mathcal{G}_f &:= \{h \in \mathcal{E}_1 \mid dh \in \langle dp, dq \rangle_{f^*\mathcal{E}_2}\}, \end{aligned}$$

where $f^* : \mathcal{E}_2 \to \mathcal{E}_1$ is the induced homomorphism between algebras of C^∞ function-germs by f. Then we have:

$$\mathcal{E}_1 \supset \mathcal{R}_f \supset \mathcal{G}_f \supset f^*\mathcal{E}_2.$$

Then in addition to that

$$\begin{aligned} \text{sp-codim}(f) &= \dim_{\mathbb{R}} \mathcal{E}_1/f^*\mathcal{E}_2 \ (= \delta(f)), \\ \text{diff-codim}(f) &= \dim_{\mathbb{R}} \mathcal{E}_1/\mathcal{G}_f, \end{aligned}$$

which appeared already, we have the following result

Theorem. (Ishikawa-Janeczko (2026)). For an $\mathcal{A}$-finite symplectic plane frontal $f : (\mathbb{R}, 0) \to \mathbb{R}^2$, we have

$$\begin{aligned} \text{fr-sp-codim}(f) &:= \dim_{\mathbb{R}} V_f^{\text{fr}}/(tf(V_1) + wf(VH_2)) \\ &= \dim_{\mathbb{R}} \mathcal{R}_f/f^*\mathcal{E}_2, \\ \text{fr-codim}(f) &:= \dim_{\mathbb{R}} V_f^{\text{fr}}/(tf(V_1) + wf(V_2)) \\ &= \dim_{\mathbb{R}} \mathcal{R}_f/\mathcal{G}_2. \end{aligned}$$

Definition. The frontal symplectic ghost number or frontal symplectic defect of f is defined by

$$\text{fsd}(f) := \text{fr-sp-codim}(f) - \text{fr-codim}(f).$$

Then we have

Corollary. fr-sp-codim(f), fr-codim(f) and fsd(f) are invariants under diffeomorphism equivalence. Moreover the frontal symplectic ghost number is equal to the symplectic ghost number of f:

$$\mathrm{fsd}(f) = \mathrm{sd}(f) = \dim_{\mathbb{R}} \mathcal{G}_2/f^*\mathcal{E}_2.$$

Example. A_4: symplectic (differential) normal form (t^2, t^5),
sp-codim = diff-codim = 2, sd = 0,
sym. (diff.) versal deformation $(t^2, t^5 + \lambda_1 t^3 + \lambda_2 t)$,
fr-sp-codim = fr-codim = 1, fsd = sd = 0.
frontal sym. versal deformation $(t^2, t^5 + \lambda_1 t^3)$.

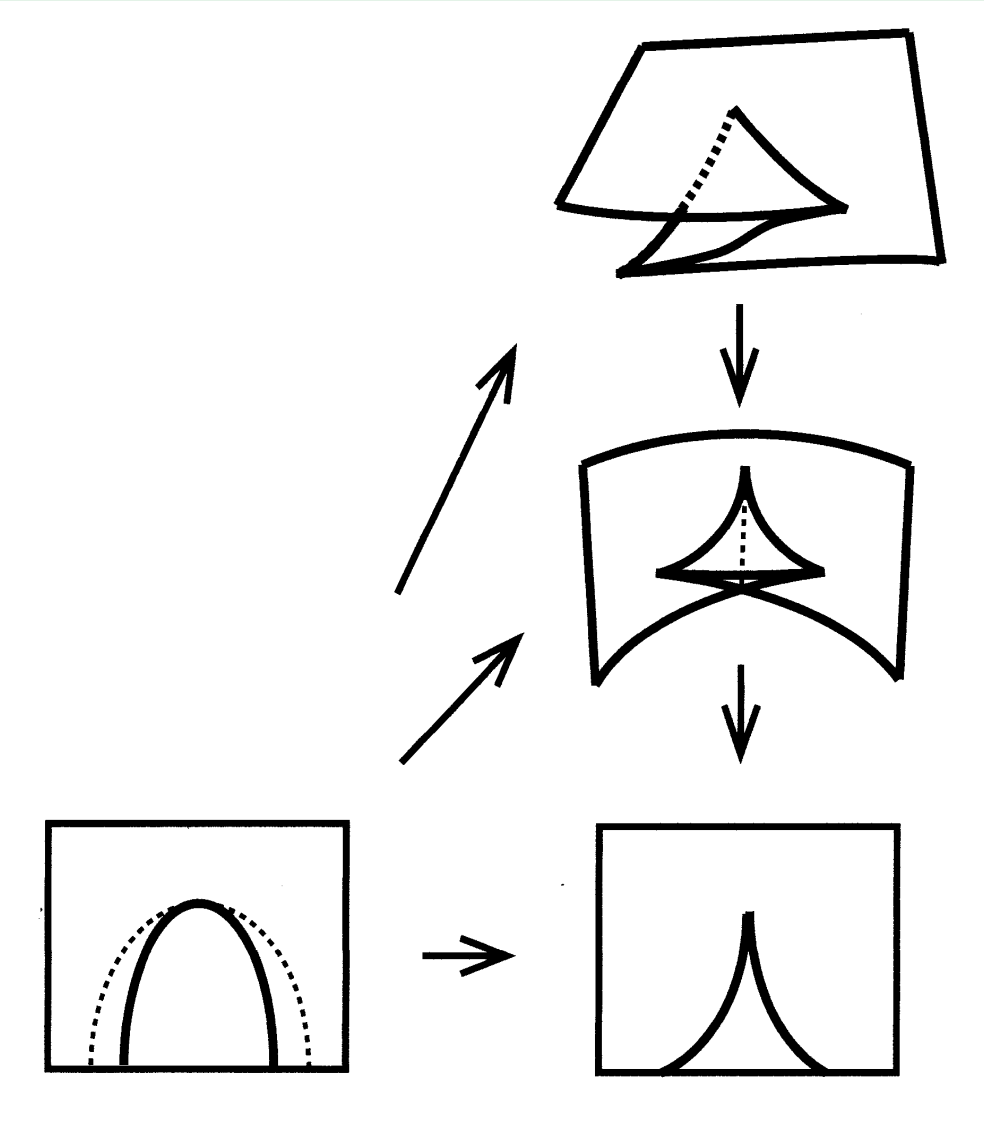
Example. E_6: symplectic (differential) normal form (t^3, t^4),
sp-codim = diff-codim = 3, sd = 0,
sym. (diff.) versal deformation $(t^3 + \lambda t, t^4 + \mu_1 t + \nu_1 t^2)$.
fr-sp-codim = fr-codim = 1, fsd = sd = 0.
frontal sym. versal deformation $(t^3 + \lambda t, t^4 + \frac{2}{3}\lambda t^2)$.

Example. W_{12}: symplectic normal form $(t^4, t^5 + \lambda t^7)$,
differential normal form $(t^4, t^5 \pm t^7), (t^4, t - 5)$,
sp-codim = 6, 6, diff-codim = 5, 6, sd = 1, 0.
symplectic versal deformation:
$(t^4 + \lambda_1 t^2 + \lambda_2 t,\ t^5 + \mu_1 t^7 + \mu_2 t^3 + \mu_3 t^2 + \mu_4 t)$.
fr-sp-codim = 3, 3, fr-codim = 2, 3, fsd = sd = 1, 0.
frontal sym. versal deformation:
$(t^4 + \lambda_1 t^2 + \lambda_2 t,\ t^5 + \mu_1 t^7 + \frac{5}{6}\lambda_1 t^3 + \frac{5}{8}\lambda_2 t^2)$.

【 Another motivation — Open swallowtails 】

V.I Arnol'd, A.B. Givental' and S. Janeczko discovered and studied the singularity, so called an open swallowtail, which is a Lagrangian variety in a space of polynomials of one variable with a symplectic structure, around 1980's. The open swallowtail appears as a generic singularity in the context of the study on shortest paths or geodesics on Riemannian manifolds with boundary.

【 Open swallowtails 】



Open swallowtail

Swallowtail

Whitney's cusp

【 Previous related researches 】

— V.I. Arnol'd, *Lagrangian manifolds with singularities, asymptotic rays and the open swallowtail,* Funk. Anal, Prilo. (1981).

— S. Janeczko *On stability of constrained Lagrangian submanifolds,* Bull. Polish Acad. Sci., (1983).

Generating families for images of Lagrangian submanifolds and open swallowtails, Math. Proc. Camb. Philo. Soc. (1986).

— A.B. Givental', *Singular Lagrangian manifolds and their Lagrangian maps,* Itogi Nauki i Tekhniki, Seriya Sovermennye Problemy Matematiki, Noveishie Dostizheniya, (1988).

— A.B. Givental', *Whitney singularities of solutions of partial differential equations,* J. of Geom. Physices, 1995.

— G. Ishikawa, (1983 Ann. Inst. Fourier), 1992 Trans. A.M.S., / 1996 Invent. math., / 2005 Asian J. of Math., / 2020 J. of Sing..

【 Open swallowtail is a Lagrangian frontal 】

We observe that the open swallowtail is a frontal which is embedded as a Lagrangian variety in a symplectic space, which we call a Lagrangian frontal.

Then we are led naturally to

Problem: Understand the open swallowtail as a "versal" or a "stable" object in the framework of singularity theory of Lagrangian frontals.

【 Versality and stability of open swallowtails 】

We give an answer to the above problem:

Theorem. (Ishikawa-Janeczko (2026))
An n-dimensional open swallowtail is realised, in fact, as the Lagrangian lift of a versal frontal deformation $\mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^2$ of the symplectic plane curve $t \mapsto (t^{k+1}, \frac{k+1}{k+2}t^{k+2})$ for some k and it is symplectically stable in Lagrangian frontals.

More details.

【 Legendrian lifts of a plane Lagrangian frontal 】

Let $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, \omega)$ be a Lagrangian frontal on the symplectic plane $\mathbb{R}^2 = \{(p, q)\}, \omega = dp \wedge dq$.

By using the symplectomorphism $(p, q) \mapsto (q, -p)$ if necessary, we suppose the tangent line $\widetilde{f}(0) \subset T_{f(0)}\mathbb{R}^2$ projects isomorphically by the projection $(p, q) \mapsto q$.

Then there exists a smooth function $r = r(t)$, the "gradient" of the line $\widetilde{f}(t)$, which satisfies $dp - rdq = 0$. Then (f, r) is a Legendrian lift of the frontal f (induced from $\widetilde{f}$) to the contact 3-space $\mathbb{R}^3 = \{(p, q, r)\}$ with the contact form $dp - rdq$.

【 Deformations and unfoldings of plane frontals 】

Let $f : (\mathbb{R}, 0) \to \mathbb{R}^2$ be a frontal.

An ℓ-parameter smooth deformation $F : (\mathbb{R}^\ell \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$, $F(u,t) = f_u(t) = (p_u(t), q_u(t))$ with $f_0 = f$ is a frontal deformation of f if there exists a smooth deformation $r_u(t)$ of $r(t)$ satisfying that $dp_u - r_u dq_u = 0$. Here we impose that $r_u(t)$ is smooth both on u and t.

We call, then, $(u, F) : (\mathbb{R}^\ell \times \mathbb{R}, 0) \to \mathbb{R}^\ell \times \mathbb{R}^2$ an ℓ-parameter frontal unfolding of f.

【Lagrangian frontal lifts of unfoldings of plane frontals】

Proposition. Let $(u, F) : (\mathbb{R}^\ell \times \mathbb{R}, 0) \to \mathbb{R}^\ell \times \mathbb{R}^2$ be a frontal unfolding of a plane frontal f.

Then there exists a unique Lagrangian frontal lift

$$\mathbb{F} : (\mathbb{R}^\ell \times \mathbb{R}, 0) \to \mathbb{R}^{2(\ell+1)} = T^*(\mathbb{R}^{\ell+1}),$$

of (u, F) with respect to the coisotropic fibration $T^*(\mathbb{R}^{\ell+1}) \to \mathbb{R}^\ell \times \mathbb{R}^2 = \mathbb{R}^\ell \times T^*\mathbb{R}$, such that

$$q_i \circ \mathbb{F} = u_i (1 \leq i \leq \ell),\ q_{\ell+1} \circ \mathbb{F} = q \circ F,\ p_{\ell+1} \circ \mathbb{F} = p \circ F$$

up to Lagrangian equivalence, with respect to the symplectic form on $T^*(\mathbb{R}^{\ell+1})$, $\Omega = \sum_{i=1}^{\ell+1} dp_i \wedge dq_i$.

【Lagrangian frontal lifts of unfoldings of plane frontals】

In fact, writing $p = p \circ \mathbb{F}, q = q \circ \mathbb{F}$, try to find

$$\mathbb{F} = \begin{pmatrix} p_1, \ldots, p_\ell, p(u,t) \\ u_1, \ldots, u_\ell, q(u,t) \end{pmatrix} : (\mathbb{R}^\ell \times \mathbb{R}, 0) \to \mathbb{R}^{2(\ell+1)} \text{ s.t.} \mathbb{F}^*\Omega = 0.$$

To do this, we set

$p_1 du_1 + \cdots + p_\ell du_\ell + p dq = dE$, for $\exists E = E(u,t)$, which is equivalent to that

$$\frac{\partial E}{\partial u_i} = p_i + p\frac{\partial q}{\partial u_i}, (1 \leq i \leq \ell), \quad \frac{\partial E}{\partial t} = p\frac{\partial q}{\partial t}.$$

Then we find, modulo functions on $u_1, \ldots, u_\ell$, that

$$\begin{aligned} E &\equiv \int p\frac{\partial q}{\partial t} dt \\ p_i &\equiv \frac{\partial E}{\partial u_i} - p\frac{\partial q}{\partial u_i} \equiv \int \frac{\partial}{\partial u_i}(p\frac{\partial q}{\partial t})dt - p\frac{\partial q}{\partial u_i} \\ &\equiv \int \frac{\partial p}{\partial u_i}\frac{\partial q}{\partial t} dt + \int p\frac{\partial^2 q}{\partial u_i \partial t} dt - p\frac{\partial q}{\partial u_i} \\ &\equiv \int (\frac{\partial p}{\partial u_i}\frac{\partial q}{\partial t} - \frac{\partial p}{\partial t}\frac{\partial q}{\partial u_i})dt. \end{aligned}$$

□

【 Examples of Lagrangian frontals ($\ell = 1$) 】

Lagrangian cuspidal edge I

$(\mathbb{R}^2, 0) \to \mathbb{R}^4, (u, t) \mapsto (p_1, p_2, q_1, q_2)$ defined by

$$p_1 = 0,\ p_2 = t^3,\ q_1 = u,\ q_2 = t^2.$$

Lagrangian cuspidal edge II

$(\mathbb{R}^2, 0) \to \mathbb{R}^4$ defined by

$$p_1 = -\tfrac{1}{2}t^4,\ p_2 = t^2,\ q_1 = u,\ q_2 = t^3 + ut^2.$$

Lagrangian open swallowtail

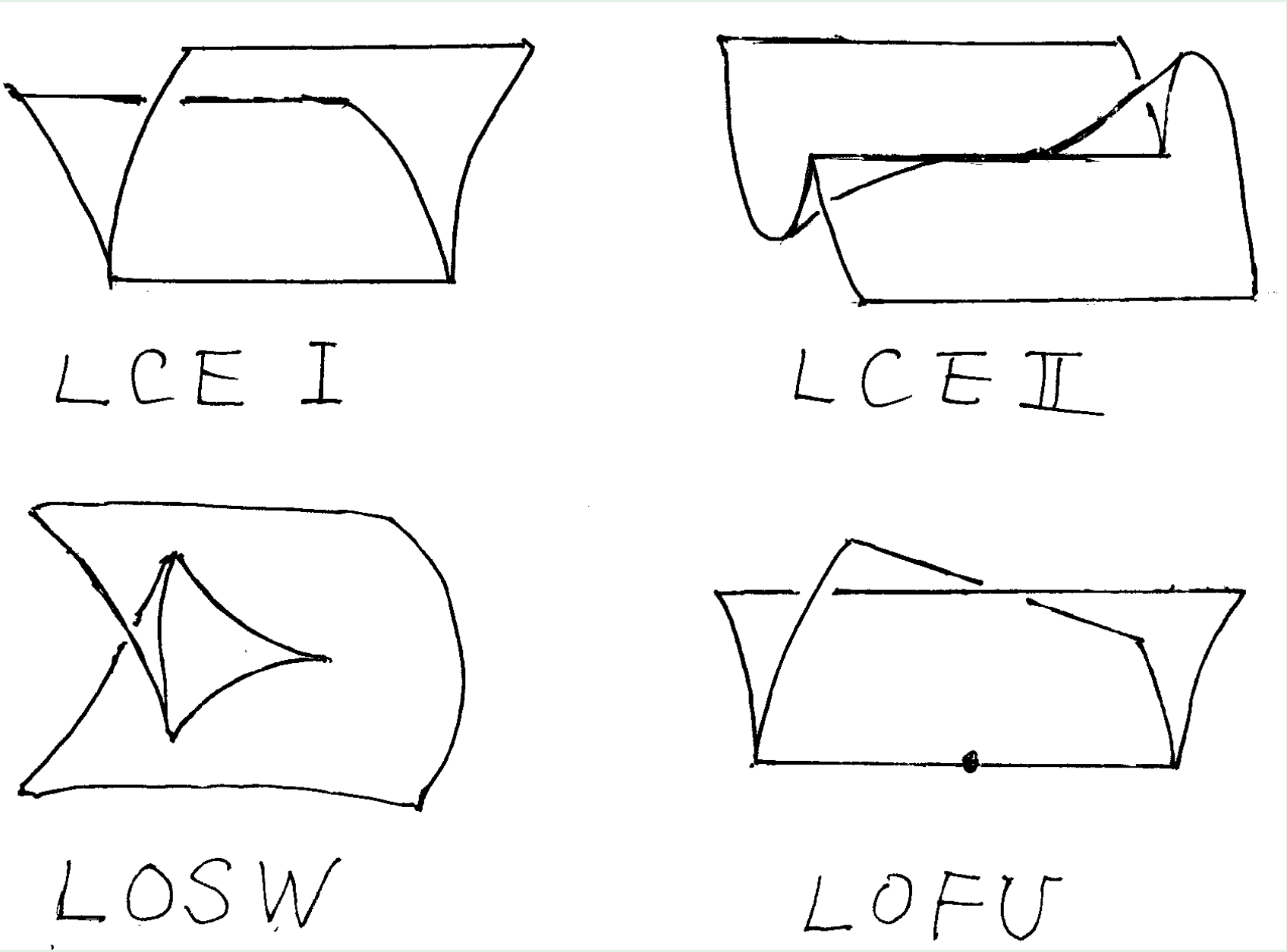
$(\mathbb{R}^2, 0) \to \mathbb{R}^4$ defined by

$$p_1 = -(\tfrac{3}{10}t^5 + \tfrac{1}{6}ut^3),\ p_2 = \tfrac{3}{4}t^4 + \tfrac{1}{2}ut^2,\ q_1 = u,\ q_2 = t^3 + ut.$$

Lagrangian open folded umbrella

$(\mathbb{R}^2, 0) \to \mathbb{R}^4$ be a map-germ defined by

$$p_1 = -\tfrac{3}{5}t^5,\ p_2 = t^5 + ut^3,\ q_1 = u,\ q_2 = t^2.$$


LCE I
LCE II
LOSW
LOFU

【 Symplectically stable Lagrangian frontals 】

Theorem. (Ishikawa-Janeczko). Let $\mathbb{F} : (\mathbb{R}^n, 0) \to \mathbb{R}^{2n} = T^*\mathbb{R}^n$ be a Lagrangian frontal of corank ≤ 1. Then the following conditions are equivalent to each other:

(i) $\mathbb{F}$ is symplectically equivalent to the Lagrangian frontal lifting of a symplectically versal frontal unfolding for a symplectic plane frontal.

(ii) $\mathbb{F}$ is symplectically stable in Lagrangian frontals.

【 Versality and stability 】

A frontal deformation $F : (\mathbb{R}^\ell \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0), F = (f_u)$ of $f_0 = f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ is called symplectically versal if, for any frontal deformation $G : (\mathbb{R}^m \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0), G = (g_v)$ of f, $g_0 = f$, there exists a map-germ $\varphi : (\mathbb{R}^m, 0) \to (\mathbb{R}^\ell, 0)$ such that g_v is symplectically equivalent to the pulled-back $f_{\varphi(v)}$ by symplectomorphisms τ_v on $T^*\mathbb{R}$ and diffeomorphisms σ_v on $\mathbb{R}$ so that $\tau_v \circ f_{\varphi(v)} \circ \sigma_v^{-1} = g_v$, $v \in (\mathbb{R}^m, 0)$.

A Lagrangian frontal-germ $\mathbb{F} : (\mathbb{R}^n, 0) \to \mathbb{R}^{2n} = T^*\mathbb{R}^n$ is called symplectically stable in Lagrangian frontals if any Lagrangian frontal deformation $\mathbb{F}_s$ of $\mathbb{F}$ is locally trivialised by symplectomorphisms T_s on $T^*\mathbb{R}^n$ and diffeomorphisms Σ_s on $\mathbb{R}^n$ so that $T_s \circ \mathbb{F}_s \circ \Sigma_s^{-1} = \mathbb{F}, (s \in (\mathbb{R}, 0)$.

【 Lagrangian open swallowtails 】

A Lagrangian open swallowtail of type k on $\mathbb{R}^n (0 \leq k \leq n)$ is given by

$$f_{n,k} : (\mathbb{R}^n, 0) \to \mathbb{R}^{2n} = T^*\mathbb{R}^n, \ f_{n,k} = f_{n,k}(u_1, \ldots, u_{n-1}, t),$$

$$\begin{aligned} f_{n,k} &= \begin{pmatrix} p_1, & \ldots, & p_{k-1}, & p_k, & \ldots, & p_{n-1}, & p_n \\ q_1, & \ldots, & q_{k-1}, & q_k, & \ldots, & q_{n-1}, & q_n \end{pmatrix} \circ f_{n,k} \\ &= \begin{pmatrix} \varphi_1, & \ldots, & \varphi_{k-1}, & 0, & \ldots, & 0, & \varphi_n \\ u_1, & \ldots, & u_{k-1}, & u_k, & \ldots, & u_{n-1}, & h \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} h &:= t^{k+1} + u_1 t^{k-1} + \cdots + u_{k-1} t, \\ \varphi_n &:= \int_0^t \tau \frac{\partial h(u,\tau)}{\partial \tau} d\tau \\ \varphi_i &:= -\tfrac{1}{k+1-i} \int_0^t \tau^{k+1-i} \frac{h(u,\tau)}{\partial \tau} d\tau, \quad (1 \leq i \leq k-1), \end{aligned}$$

and $\varphi_j = 0 \ (k \leq j \leq n-1)$.

【 Lagrangian open swallowtails 】

We give an answer to the above mentioned problem:

Theorem. (Ishikawa-Janeczko)

The Lagrangian open swallowtail of type k

$$f_{n,k} : (\mathbb{R}^n, 0) \to \mathbb{R}^{2n} = T^*\mathbb{R}^n, (0 \leq k \leq n),$$

is the Lagrangian lift of a versal frontal deformation of the symplectic plane curve $t \mapsto (t^{k+1}, \frac{k+1}{k+2}t^{k+2})$ and it is symplectically stable in Lagrangian frontals.

Dziękuję za uwagę !

Thank you for your attention !