FOURIER TRANSFORM and ANALYTIC CONTINUATION of POWER CONSTRUCTIBLE FUNCTIONS

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General Goal

Find a "rich" algebra of functions $f : \mathbb{R} \to \mathbb{C}$, built on a class of **tame functions** where one can study the interplay between

- the **Fourier transforms** of analytic L^1 functions,
- and their **complex analytic extension to** $\mathbb{C}$.

1- Introduction: our 4 algebras

Let $\mathbb{K} \subset \mathbb{C}$ a field extension, $n \in \mathbb{N}$ and $X \subset \mathbb{R}^n$ a subanalytic set.

- $\mathcal{S}(X)$ the $\mathbb{C}$-algebra of (globally) subanalytic functions on X.
- $\mathcal{C}^{\mathbb{K}}(X)$: the $\mathbb{C}$-algebra of **power constructible functions**

$$\boxed{g \in \mathcal{C}^{\mathbb{K}}(X) \Longleftrightarrow g = P\left(f_1^{k_1}, \ldots, f_\ell^{k_\ell}, \log(f_{\ell+1}), \ldots, \log(f_p)\right)}$$

$P \in \mathbb{C}[X_1, \ldots, X_{\ell+p}], \quad f_i \in \mathcal{S}(X), \quad k_i \in \mathbb{K}.$

- $\mathcal{C}^{\mathbb{K},\mathrm{i}\mathcal{S}}(X) = \mathcal{C}^{\mathbb{K}}(X)[\mathrm{e}^{\mathrm{i}f}, f \in \mathcal{S}(X)]$.
- $\mathcal{C}^{\mathbb{K},\mathcal{F}}(X)$ $\mathbb{C}$-algebra of functions

$$X \ni x \mapsto \int_{t\in\mathbb{R}} g(x,t)\mathrm{e}^{\mathrm{i}t}\ \mathrm{d}t, \quad g \in \mathcal{C}^{\mathbb{K}}.$$

1- Introduction: Stability under integral transfroms

$$\boxed{\mathcal{S} \subsetneq \mathcal{C}^{\mathbb{K}} \subsetneq \mathcal{C}^{\mathbb{K},\mathrm{i}\mathcal{S}} \subsetneq \mathcal{C}^{\mathbb{K},\mathcal{F}}}$$

- $\mathcal{C}^{\mathbb{R}} \subset \mathbb{R}_{\mathrm{an,exp}}$, but in case $\mathrm{i} \in \mathbb{K}$, $\sin(\log) \in \mathcal{C}^{\mathbb{K}} \not\subset \mathbb{R}_{\mathrm{an,exp}}$.
- $\mathcal{C}^{\mathbb{K}}$ is the smallest algebra containing the $\mathbb{K}$-powers of functions of $\mathcal{S}$ and stable under **parametric integration**:

$$x \mapsto \int_{t\in\mathbb{R}} f(x,t)\ \mathrm{d}t$$

- $\mathcal{C}^{\mathbb{K},\mathcal{F}}$ is the smallest algebra containing the $\mathbb{K}$-powers of functions of $\mathcal{S}$ and stable under **parametric Fourier transform**:

$$\xi \mapsto \mathcal{F}(f)(x,\xi) := \int_{t\in\mathbb{R}} f(x,t)\mathrm{e}^{-\mathrm{i}2\pi\xi t}\ \mathrm{d}t$$

  $\mathcal{C}^{\mathbb{K},\mathcal{F}}$ is also stable under parametric integration (the smallest such algebra containing $\mathcal{C}^{\mathbb{K},\mathrm{i}\mathcal{S}}$).

  $\mathcal{C}^{\mathbb{K},\mathcal{F}}$ is very large : it contains smooth functions with compact support.

1- Introduction: Stability under integral transfroms

Key points:

$\mathcal{C}^{\mathbb{K}}$ is an algebra:

- **containing** $f, \log f, f^k$, with f subanalytic and $k \in \mathbb{K}$,
- stable by integration.

- **contained** in two auxiliary algebras $\mathcal{C}^{\mathbb{K},\mathrm{i}\mathcal{S}} \subset \mathcal{C}^{\mathbb{K},\mathcal{F}}$, with $\mathcal{C}^{\mathbb{K},\mathcal{F}}$ stable by integration and Fourier transform.

1- Introduction: More precise Goal

- Study the decay of $\xi \mapsto \mathcal{F}(f)(\xi)$ for $f \in \mathcal{C}^{\mathbb{K}}(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$.

  $\mathcal{F}(f)$ is continuous and goes to 0 at $\pm\infty$ (Riemann-Lebesgue).

  Beyond this general fact, how fast $\mathcal{F}(f)$ goes to 0?

  In particular, when is $\mathcal{F}(f)$ **decaying exponentially at** $\pm\infty$?

$$\text{i.e. } \mathcal{F}(f)(\xi) = O_{\xi\to\pm\infty}(\mathrm{e}^{-a|\xi|}),\ a > 0.$$

  Example: $f(t) = \dfrac{1}{1+t^2} \in \mathcal{S}(\mathbb{R}) \cap L^1(\mathbb{R})$ and

$$\mathcal{F}(f)(\xi) = \pi \mathrm{e}^{-2\pi|\xi|}.$$

  Here $a = 2\pi$ and $a/2\pi$ is the distance, to the real axis, of the complex poles of the meromorphic (rational) extension f to $\mathbb{C}$.

- **Question:** Is there an interplay between

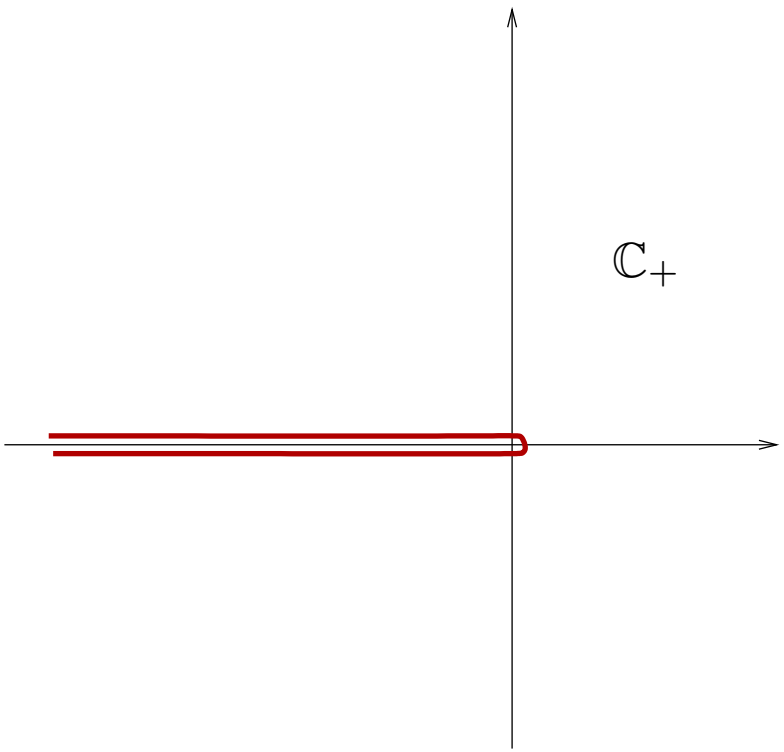
  - the **rate of the exponential decay of** $\mathcal{F}(f)$

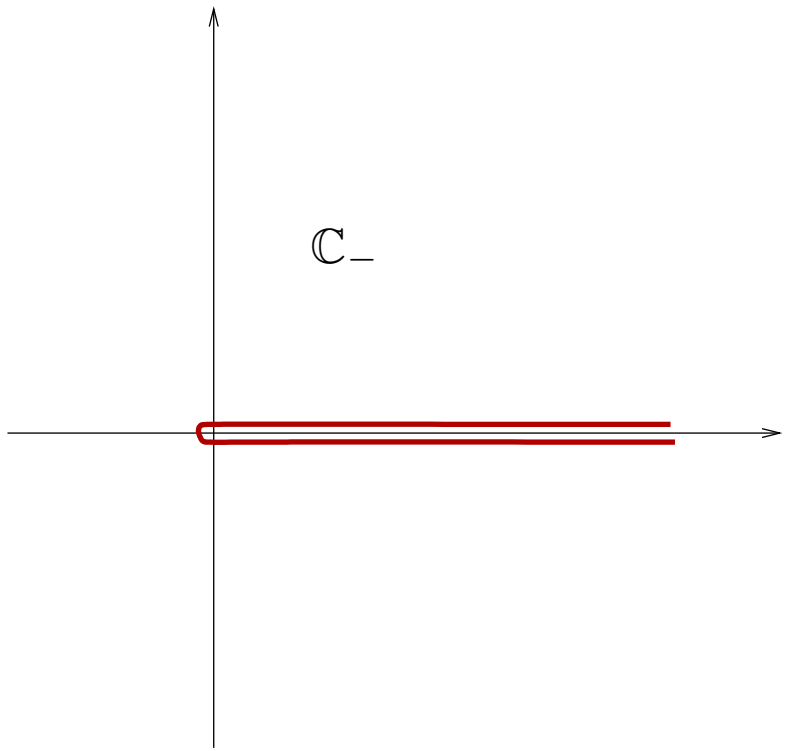
  and

  - the **"maximal" domain of** $\mathbb{C}$ **to which** f **holomorphically extends**?

2- Analytic extension to domains of $\mathbb{C}$: at ∞

Let us denote, for $\sigma \in \{-, +\}$, $\mathbb{C}_\sigma := \mathbb{C} \setminus \mathbb{R}_{-\sigma}$.





2- Analytic extension to domains of $\mathbb{C}$: at ∞

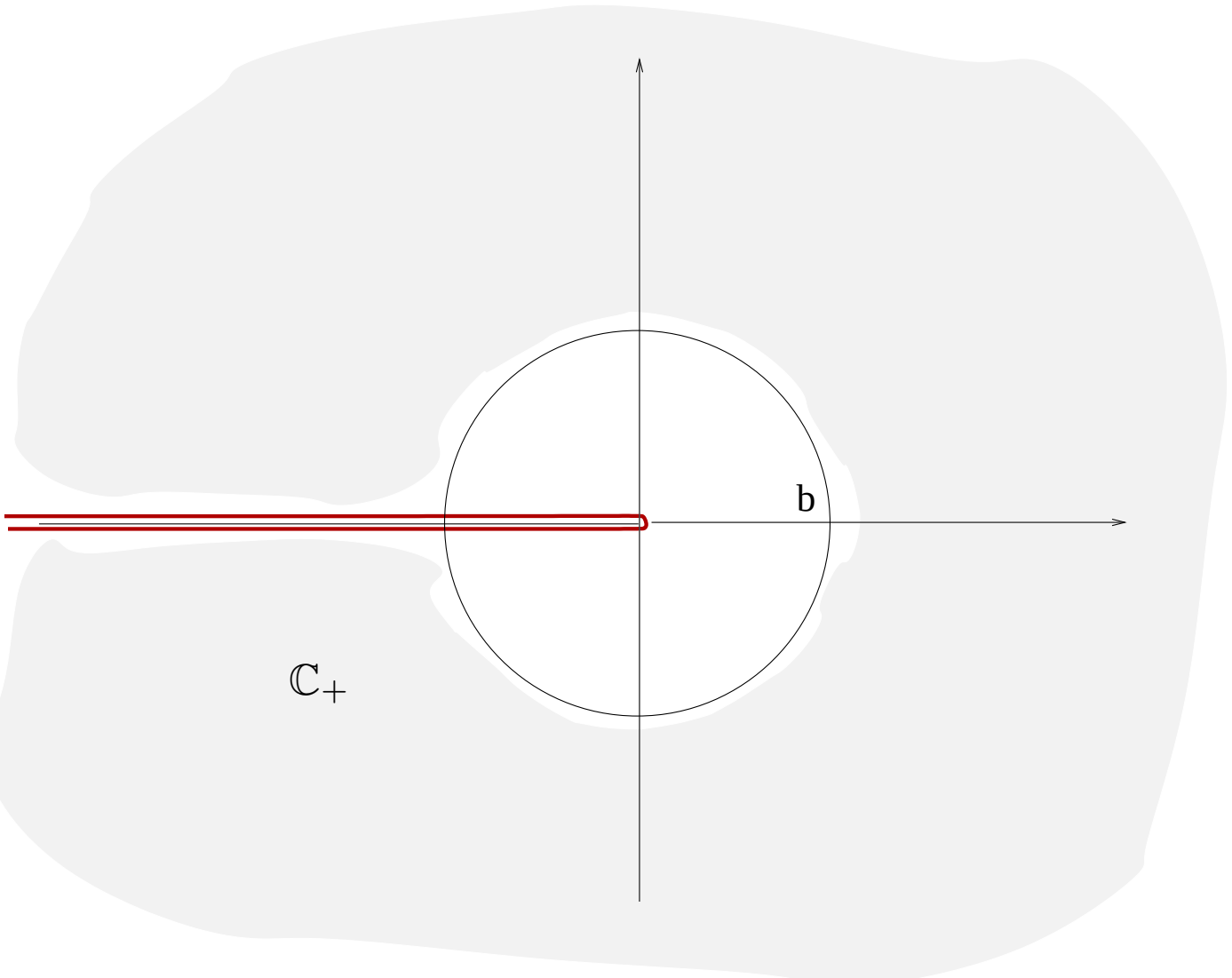
Theorem 1. *Let $\mathcal{D} = \mathcal{C}^{\mathbb{K}}$ or $\mathcal{C}^{\mathbb{K},i\mathcal{S}}$ and $f \in \mathcal{D}(\mathbb{R})$, smooth. For each $\sigma \in \{-,+\}$, there exists $b > 0$, s.t.*

f extends on $\{z \in \mathbb{C}_\sigma, |z| > b\}$ to a **holomorphic $\mathcal{D}$-function** *f_σ,* **polynomialy bounded** *when $\mathcal{D} = \mathcal{C}^{\mathbb{K}}$.*

Proof. Prepare functions of $\mathcal{D}$ using subanalytic preparation. □

Remark: The family version holds.

2- Analytic extension to domains of $\mathbb{C}$: at ∞



2- Analytic extension to domains of $\mathbb{C}$: at ∞

Compare with:

- **Theorem** (Wilkie 2016).
  $f \in \mathbb{R}_{\mathrm{an,exp}}$ has an $\mathbb{R}_{\mathrm{an,exp}}$-definable analytic continuation to a (right) half-plane of $\mathbb{C} \iff f \in \mathcal{R}$,

  where $\mathcal{R}$ is in the residue field of the Hardy field of $\mathbb{R}_{\mathrm{an,exp}}$ with valuation ring polynomially bounded germs ($\mathcal{C}^{\mathbb{R}} \subsetneq \mathcal{R}$)
  ie $\mathcal{R}$ are poly. bounded functions up to exp. flat functions.

- **Corollary of Theorem 1.** *$f \in \mathcal{C}^{\mathbb{K}}(\mathbb{R})$ extends to an entire function $\iff$ f is polynomial.*

  Proof. By Thm 1 such an extension is polynomially bounded. □

- **Remark.** $f(t) = \mathrm{e}^{\mathrm{i}t} \in \mathcal{C}^{\mathbb{K},\mathrm{i}\mathcal{S}}$ analytically extends to $\mathbb{C}$ but $f(-\mathrm{i}t) = \mathrm{e}^{t}$ is not polynomially bounded, therefore Thm 1 and its Corollary do not hold in $\mathcal{C}^{\mathbb{K},\mathrm{i}\mathcal{S}}$.

3- Analytic extension to domains of $\mathbb{C}$: horizontal strips

Theorem 2. *Let $\mathcal{D} = \mathcal{C}^{\mathbb{K}}$ or $\mathcal{C}^{\mathbb{K},i\mathcal{S}}$ and $f \in \mathcal{D}(\mathbb{R}) \cap C^\infty(\mathbb{R})$, $\sigma \in \{-,+\}$.*

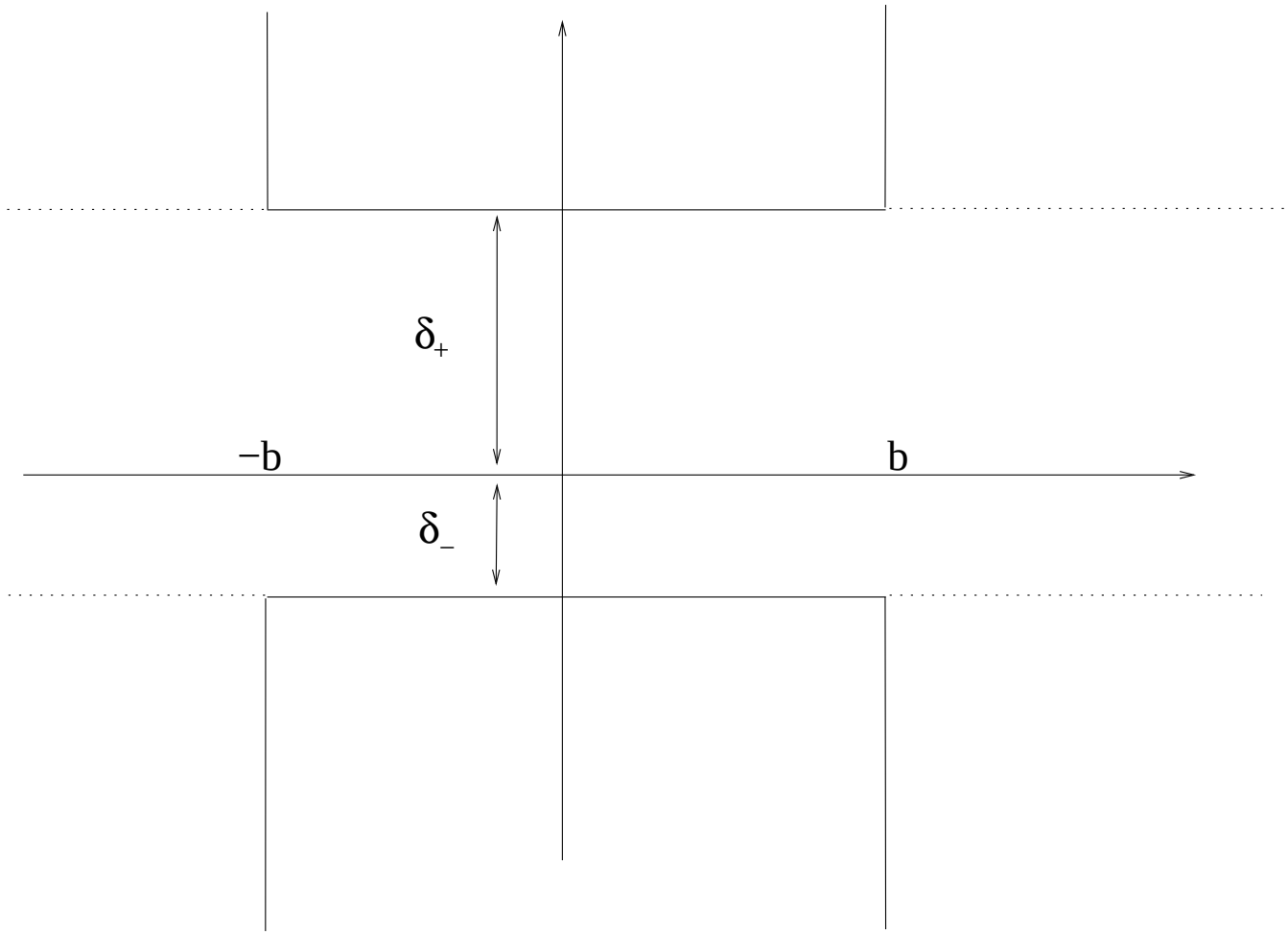
Then f extends to an **analytic** *$\mathcal{D}$-function on the (maximal) horizontal strip*
$\mathbb{R} + \sigma \mathrm{i}[0, \delta_\sigma)$.

Remark: The family version holds.

Proof. If $f \in \mathcal{D}(\mathbb{R}) \cap C^\infty(\mathbb{R})$ then f is analytic. Apply analytic extension at ∞. □

Example. $f(t) = \sum_{n\geq 1} \dfrac{1}{(t-n)^2 + 1/n^2} \notin \mathcal{C}^{\mathbb{K},\mathrm{i}\mathcal{S}}$: the poles are $n + \dfrac{i}{n}$.

3- Analytic extension to domains of $\mathbb{C}$



4- Decay of Fourier transform of functions in $\mathcal{C}^{\mathbb{K}}$

Theorem 3. *Let $f \in \mathcal{C}^{\mathbb{K}}(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$. Up to modifying a finite number of values of f*

$$\mathcal{F}(f) \text{ decays exponentially at } \pm\infty \Longleftrightarrow f \in C^{\infty}(\mathbb{R}).$$

Notation: $\eta_{\sigma}(f) := \sup\{a \geq 0; \mathcal{F}(f)(\xi) = O_{\xi\to\sigma\infty}(\mathrm{e}^{-\sigma 2\pi a\xi})\}$,

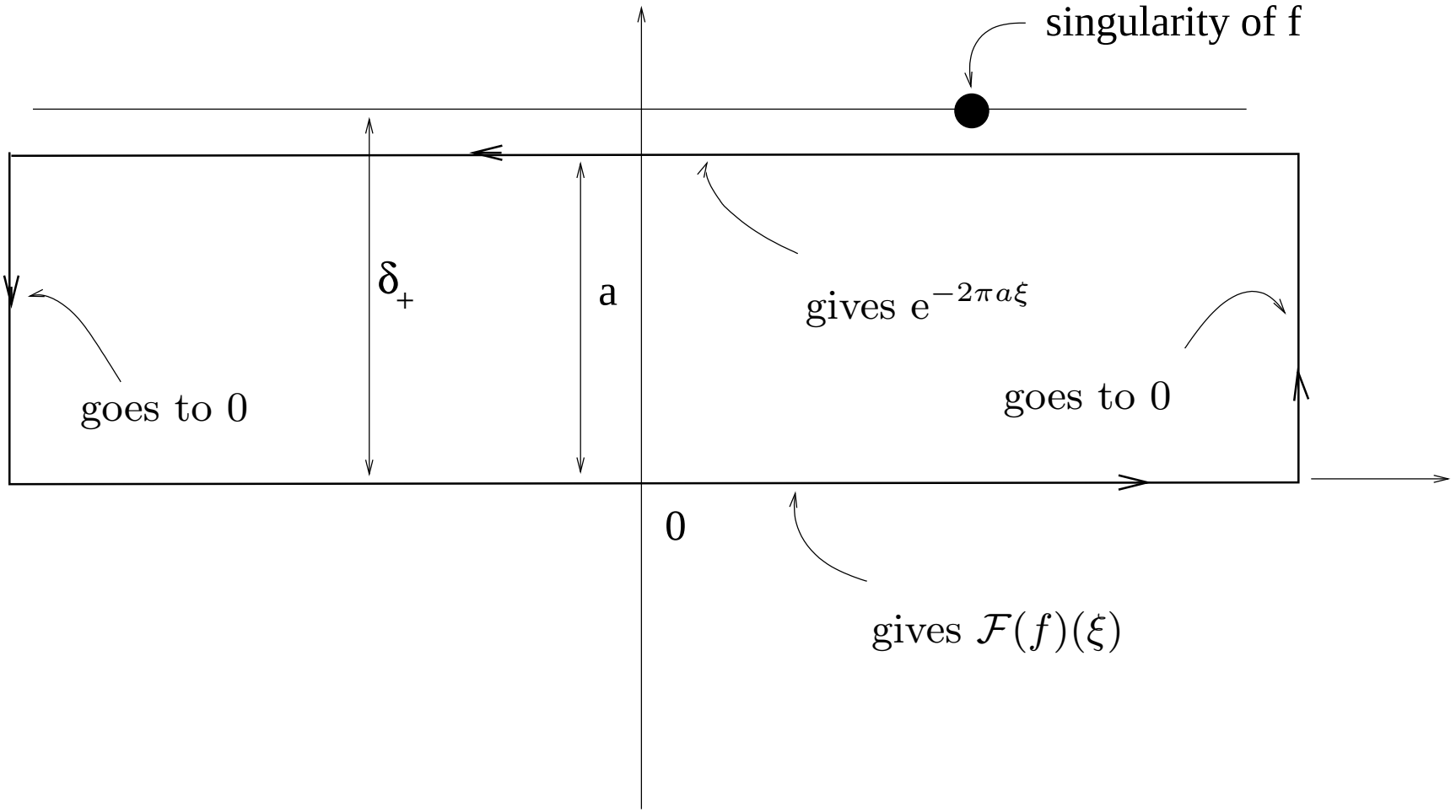
then

$$\boxed{\eta_{\sigma}(f) = \delta_{\sigma}(f).}$$

Proof.
$\Longleftarrow$: use $\widehat{\mathcal{F}}$ and holomorphy under integral sign.

$\Longrightarrow$: use extension on strips about the real axis and integration on rectangles. □

4- Decay of Fourier transform of functions in $\mathcal{C}^{\mathbb{K}}$



4- Exemple of decay of $\mathcal{F}(f)$

Remark. If $f \in \mathcal{C}^{\mathbb{K}}(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$ extends to an entire function, $\eta_\sigma(f) = \delta_\sigma(f) = +\infty$. It follows that $\mathcal{F}(f) = 0$, and thus $f = 0$.

Example. Take

$$f(t) = \frac{1}{(t - \mathrm{i})^2}, \quad f \in \mathcal{S}.$$

Then

- $\delta_+ = \eta_+ = 1$ and $\delta_- = \eta_- = +\infty$.

  Therefore $\mathcal{F}(f)(\xi) = 0$ for $\xi \geq 0$ and $\mathcal{F}(f)(\xi) = O_{\xi \to -\infty}(\mathrm{e}^{2\pi\xi})$.

- A direct computation indeed gives

$$\mathcal{F}(f)(\xi) = \begin{cases} 0, & \text{if } \xi \geq 0 \\ 4\pi^2 \xi \mathrm{e}^{2\pi\xi}, & \text{if } \xi < 0 \end{cases}$$

6- Counter-example to Theorem 3 in $\mathcal{C}^{\mathbb{Q},\mathrm{i}\mathcal{S}}$

- Take $f(t) = \dfrac{\mathrm{e}^{\mathrm{i}t^2}}{1+t^2} \in \mathcal{C}^{\mathbb{Q},\mathrm{i}\mathcal{S}}(\mathbb{R}) \setminus \mathcal{C}^{\mathbb{Q}}(\mathbb{R})$, then

$$\mathcal{F}(f)(\xi) \underset{\xi \to \sigma\infty}{\sim} \sigma C t e \frac{\mathrm{e}^{\mathrm{i}(\pi\xi)^2}}{\xi^2},$$

and $\eta_\sigma = 0 \neq 1 = \delta_\sigma$.

7- Meromorphic extension to $\mathbb{C}$

Theorem 5. *Let $f \in \mathcal{C}^{\mathbb{K}}(X \times \mathbb{R})$ s.t. $\forall x \in X$, $f_x \in C^\infty(\mathbb{R})$ and extends to a meromorphic function on $\mathbb{C}$.*

Then $\exists P \in \mathcal{C}^{\mathbb{K}}(X)[t], Q \in \mathcal{S}(X)[t]$ s.t.

$$f(x,t) = \frac{P(x,t)}{Q(x,t)},\ \forall (x,t) \in X \times \mathbb{R}.$$

Compare with (we do not assume definability):

- **Theorem** (Peterzil-Starchenko 2001). *If $f : \mathbb{C} \setminus \{w_1, \ldots, w_\ell\} \to \mathbb{C}$ is definable in some o-minimal structure and analytic, then f is rational over $\mathbb{C}$.*

Proof. -Absolute case: By Theorem 1 the poles of f are in a ball. Thus $(z - w_1) \cdots (z - w_\ell) f(z)$ is entire, and thus is a polynomial.
- Family case: work harder. □

Remark. Not true in $\mathcal{C}^{\mathbb{K},\mathrm{i}\mathcal{S}}$: Again $f(t) = \mathrm{e}^{\mathrm{i}t}$ is entire.

8- Integrability of $\mathcal{F}(f)$ for $f \in \mathcal{C}^{\mathbb{K}}$

Let $f \in \mathrm{L}^1(\mathbb{R})$, then if $\mathcal{F}(f) \in \mathrm{L}^1(\mathbb{R})$, by inverse Fourier transform, f has to be continuous (up to changing a finite number of values of f).

For $f \in \mathcal{C}^{\mathbb{K}}(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$ continuity is in fact a **sufficient conditon for integrability of** $\mathcal{F}(f)$.

Theorem 6. (see Clukers-Miller for $\mathcal{C}^{\mathbb{Q}}$) *If* $f \in \mathcal{C}^{\mathbb{K}}(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$ *is continuous, then* $\mathcal{F}(f) \in \mathrm{L}^1(\mathbb{R})$ (and $\in \mathcal{C}^{\mathbb{K},\mathcal{F}}(\mathbb{R})$).

Counter-example in $\mathcal{C}^{\mathbb{K},\mathcal{F}}$. Let $f(t) = t\mathrm{e}^{\mathrm{i}t^4/4}$. Define the improper integral function

$$g(x) := -\int_x^{+\infty} f(t)\ \mathrm{d}t.$$

- Since $\mathcal{C}^{\mathbb{K},\mathcal{F}}$ is closed under parametric integration and simple limits, $g \in \mathcal{C}^{\mathbb{K},\mathcal{F}}(\mathbb{R}) \cap C^0(\mathbb{R})$.
- Integration by parts shows that $g \in \mathrm{L}^1(\mathbb{R})$.
- $\mathcal{F}(g)(\frac{\xi}{2\pi}) \underset{\xi \to \sigma\infty}{\sim} \sigma \frac{C}{\xi} \mathrm{e}^{-\mathrm{i}\frac{3}{4}\xi^{\frac{4}{3}}} \notin \mathrm{L}^1(\mathbb{R})$.