Uniqueness in Mean Curvature Flow

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joint work with Toby Colding

Łojasiewicz 100, September 2026

Overview

- Mean curvature flow.
- Singularities and blowups.
- Uniqueness of blowups.
- Quantitative uniqueness.

Gradient descent

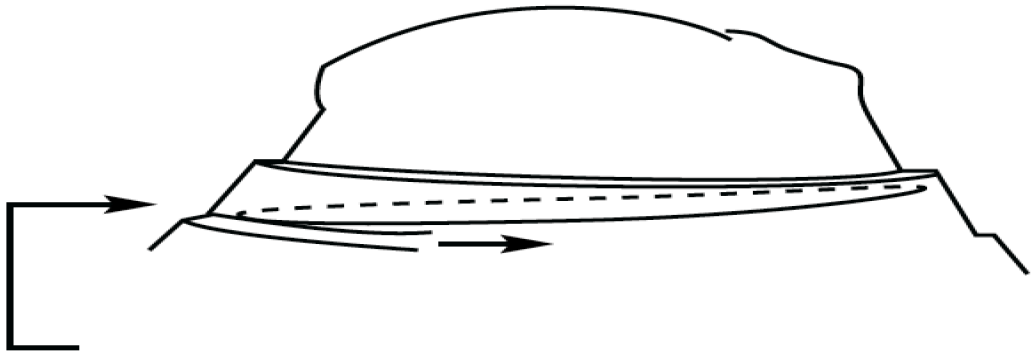
- **Goal**: Minimize a positive function $f(x)$.
- **Strategy**: Evolve $\gamma(t)$ by $\frac{d\gamma}{dt} = -\nabla f(\gamma(t))$.
- $f(\gamma(t))$ decreases

$$\frac{d}{dt} f(\gamma(t)) = -\left|\nabla f(\gamma(t))\right|^2 .$$

- $f(\gamma(t))$ has a limit as $t \to \infty$. Does $\gamma(t)$?

Uniqueness can fail for a gradient flow

Goat track example:



Narrow ledge on steep hillside, becoming more and more narrow and sloping less and less as it spirals infinitely down to the base.

The base is a circle where f vanishes to infinite order.

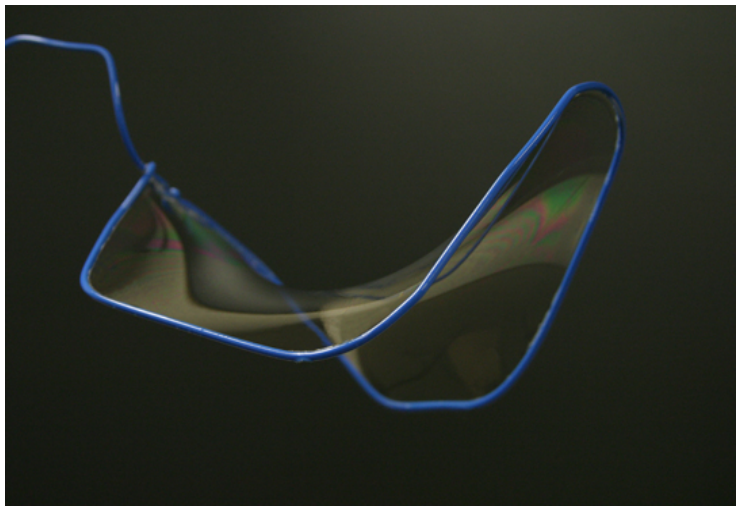
A gradient descent spirals infinitely into this circle.

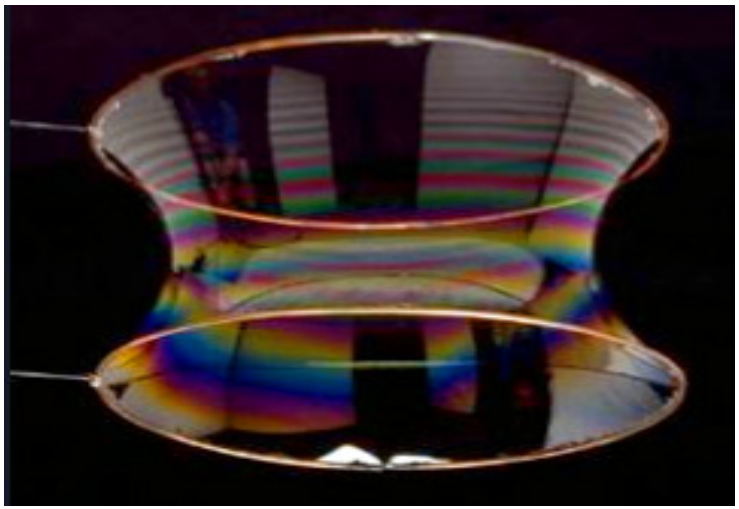
Łojasiewicz Theorem: Analytic gradient flows

- Uniqueness can fail for a general gradient flow.
- **Łojasiewicz**: Uniqueness for **analytic** gradient flows.
- Gradient Łojasiewicz inequality $\Longrightarrow$ rate of convergence.

Surface tension

The mean curvature H is the force of surface tension.





Surface tension pulls a soap film tight: $H = 0$.

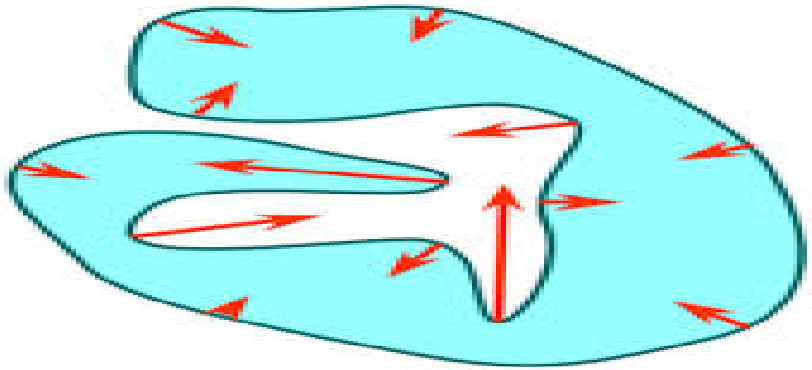
Mean curvature flow

- Mean curvature flow – submanifold pulling itself tight

$$\frac{\partial x}{\partial t} = -H\,\mathbf{n}\,.$$

- Gradient descent: Volume decreases

$$\frac{\partial}{\partial t}\,\mathrm{Vol}(\Sigma_t) = \int_{\Sigma_t} H\,\langle \frac{\partial x}{\partial t}, \mathbf{n}\rangle = -\int_{\Sigma_t} H^2\,.$$



Red arrows show how points evolve at this instant.

Avoidance

Σ_t and Γ_t smooth compact MCF's for $t \in [0, T]$.

- Avoidance principle:

  If $\Sigma_0 \cap \Gamma_0 = \emptyset$, then $\Sigma_t \cap \Gamma_t = \emptyset$ for $t > 0$.

- More: Distance between them increases.

Weak solutions

The volume decay formula and avoidance are “low regularity” properties that can be used to define weak solutions:

- Brakke’s MCF of varifolds (measure theoretic).
- Osher-Sethian, Evans-Spruck, Chen-Giga-Goto:

  Level set flow (viscosity solutions).

Shrinking spheres

MCF preserves rotational symmetry $\Longrightarrow$ ODE for the radius:

$$r'(t) = -\frac{n}{r(t)}\,.$$

- n-spheres $S^n_{\sqrt{-2nt}}$ with $r(t) = \sqrt{-2nt}$ for $t < 0$.
  - Defined for all $t < 0$ (ancient).
  - Singular at $t = 0$.
  - Self-similar: $\Sigma_t = \sqrt{-t}\,\Sigma_{-1}$.

Cylinders

Cylinders $S^k \times \mathbf{R}^{n-k}$ of radius $\sqrt{-2kt}$ for $t < 0$.

Ancient. Singular at $t = 0$; singular set is $\mathbf{R}^{n-k}$.

Self-similar: $\Sigma_t = \sqrt{-t}\,\Sigma_{-1}$.

Singularities

- Avoidance & spheres: *Any closed MCF becomes singular*.
- Analyze singularities: Blowup analysis.
- Illustrate this with examples next.

Convex hypersurfaces

Gage-Hamilton (curves), **Huisken** (higher dim):

A closed *convex* hypersurface remains convex and flows smoothly until it disappears at a point.

Increasingly round just before extinction.

The marriage ring shrinks to a circle

A thin torus of revolution disappears along a circle.



This circle is the singular set of the flow.

Flow looks cylindrical near each singularity.

Axis of limiting cylinder $=$ tangent to singular curve.

Grayson's Dumbbell

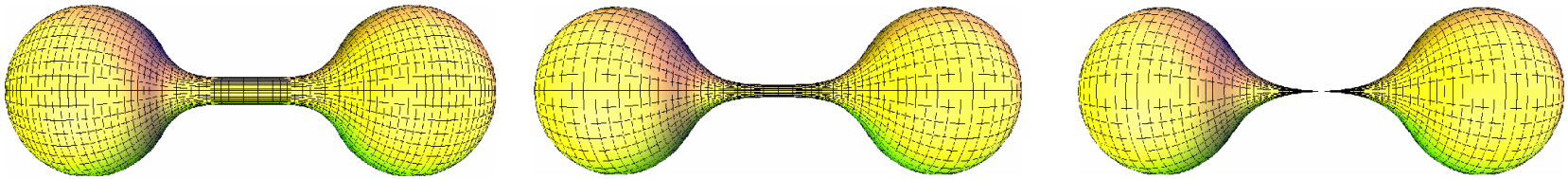


Figure: Initial dumbbell $\rightarrow$ shrinking neck $\rightarrow$ neck pinch.

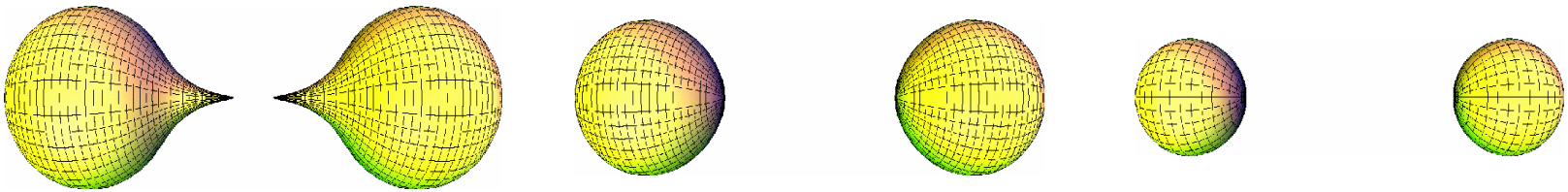
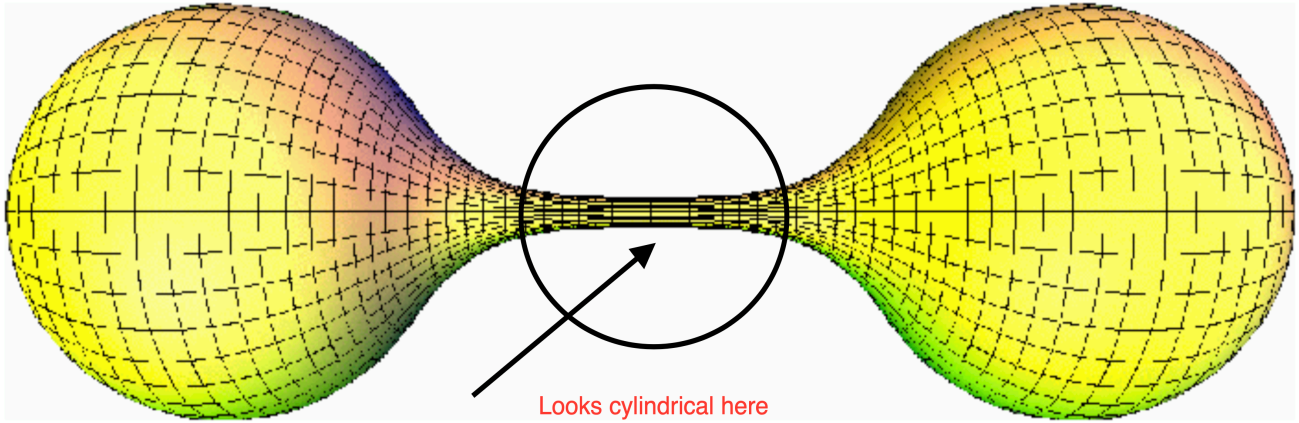


Figure: Cusps retract, pieces become round.

Blowing up the neck pinch

- Zoom in on the singularity to get sequence of MCFs.
- Closer and closer to larger pieces of cylinders.



Singularities in geometric flows

Blowup analysis

- Rescale the flow to zoom in on a singularity.
- Monotonicity (Huisken): Uniform control.
- Tangent flow (Ilmanen, White):
  Convergent subsequence, dilation invariant[1] limit.

[1] Dilation = parabolic dilation, so $\Sigma_t = \sqrt{-t}\,\Sigma_{-1}$.

MCF Shrinkers

- Tangent flows evolve by rescaling $M_t = \sqrt{-t}\, M_{-1}$.
- Shrinker (M_{-1}) satisfies $\mathbf{H} = \frac{x^\perp}{2}$.

  Think: $\mathbf{H} + \frac{x^T}{2} = \frac{x}{2}$ (**H** plus tangential gives scaling).
- Euler-Lagrange equation for Gaussian area

$$F(\Sigma) = (4\pi)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|x|^2}{4}}.$$

Uniqueness of blowups question

- Compactness gave a convergent subsequence.
- Different limits for different subsequences?

Uniqueness for spherical blowups

Huisken implies uniqueness at spherical singularities.

A priori, uniqueness might fail two ways at a cylinder:

- An entirely different shrinker.
- A cylinder, but rotated.

Uniqueness as a dynamical question

- We can reformulate uniqueness as a dynamical question for an infinite dimensional gradient flow.
- The flow comes from minimizing gaussian area F.
- This is called rescaled MCF.

Uniqueness and rescaled MCF

- Blowing up is equivalent to rescaled MCF

$$(\star) \qquad \frac{\partial x}{\partial t} = \left(\frac{1}{2} \langle x, \mathbf{n} \rangle - H \right) \mathbf{n} \, .$$

  Red term generates scaling, up to tangential diffeos.

- Fixed points of $(\star)$ are critical points of F, i.e., shrinkers.

- $(\star)$ is gradient descent for gaussian area F

  $\Longrightarrow$ $F(\Sigma_t)$ is decreasing, has limit as $t \to \infty$.

General uniqueness theorems

Two general cases where uniqueness holds:

- non-degenerate (Morse) or analytic (Łojasiewicz).
- There are infinite dimensional analogs:
  - Łojasiewicz-Simon: Lyapunov-Schmidt reduction.
  - Allard-Almgren: Integrability.

Uniqueness of blowups

Theorem, CM. *Cylindrical MCF blowups are unique.*

- Covers all mean convex flows.
  (White, Huisken-Sinestrari, cf. Andrews, Haslhofer-Kleiner)
- Covers all generic singularities in $\mathbf{R}^3$
  (CM, Chodosh-Choi-Mantoulidis-Schulze, Bamler-Kleiner).

Conical uniqueness: Chodosh-Schulze, cf. T.-K. Lee–X. Zhao.

Rectifiability of $\mathcal{S}$

Uniqueness key for sharp regularity:

- $\Sigma_t^2 \subset \mathbf{R}^3$ mean convex.
- **CM**: $\mathcal{S}$ is contained in a finite union of compact C^1 curves plus a countable set of points.
- Corollary: $\mathcal{H}_1(\mathcal{S}) < \infty$ (sharp on marriage ring).
- White showed that $\mathcal{S}$ is at most one dimensional.

Outline

1 Mean curvature flow

2 Singularities and blowups

3 Uniqueness of blowups
- Quantitative uniqueness
- Strong rigidity

Quantitative uniqueness

Rescaled MCF Σ_t for $t \in [t_1, t_2]$. Suppose that:

- Σ_{t_1} is close to a cylinder $\mathcal{C}$, and
- $F(\Sigma_t)$ is almost constant on $[t_1, t_2]$.

Theorem, CM: Σ_{t_1} and Σ_{t_2} are close.

Key: independent of the time $(t_2 - t_1)$ flowed.

Łojasiewicz: true for *finite dimensional* analytic gradient flows.

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The quantitative uniqueness theorem

Theorem, CM: If a rescaled MCF Σ_t has

1. $\text{dist}\,(\Sigma_{t_1}, \mathcal{C}) < \epsilon_1$, and
2. $|F(\Sigma_{t_i}) - F(\mathcal{C})| < \epsilon_2$ for $i = 1, 2$,

then $\text{dist}\,(\Sigma_t, \Sigma_{t_1}) < c \sum_{i=1,2} |F(\Sigma_{t_i}) - F(\mathcal{C})|^\alpha$.

$\alpha, \epsilon_1, \epsilon_2 > 0$ and c depend on n and entropy λ.

This implies uniqueness.

Holds in a manifold, using Hirsch and J.J. Zhu (2025).

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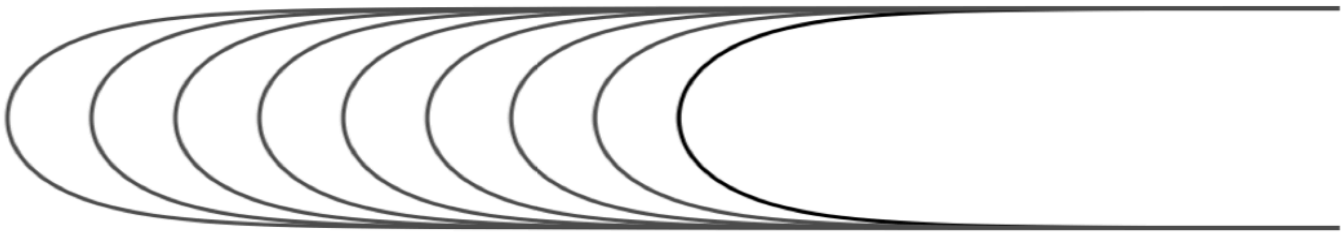
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Blow downs

- Ancient MCF M_t. Blow down $\overline{\Sigma}_t$ is a limit of

$$M^i_t = \lambda_i^{-1} \, M_{\lambda_i^2 \, t} \text{ with } \lambda_i \to \infty \, .$$

- Grim reaper $x = t - \log \cos y$ translates across $\mathbf{R}^2$.



Dilating squashes it and speeds it up.

$\Longrightarrow$ $\overline{\Sigma}_t$ is the line (multiplicity two) for $t < 0$.

Application: Blow downs

Ancient MCF M_t with bounded entropy λ.

- Blow down $\overline{\Sigma}_t$ is a limit of

$$M^i_t = \lambda_i^{-1} M_{\lambda_i^2 t} \text{ with } \lambda_i \to \infty .$$

- Monotonicity: $\overline{\Sigma}_t$ is self-shrinking.
- Quantitative uniqueness implies that:

  If one $\overline{\Sigma}_t$ is a cylinder, every $\overline{\Sigma}_t$ is the same cylinder.

Application: General tangent flows

MCF M_t with cylindrical singularity $\mathcal{C}$ at $(0,0)$.

- Suppose $x_i \to 0$, $s_i \to 0$, $\mu_i \to 0$ and

$$\frac{1}{\mu_i}\left(M_{s_i-\mu_i^2} - x_i\right) \to \mathcal{O}(\mathcal{C})$$

where $\mathcal{O}$ is a rotation in $\mathbf{R}^{n+1}$.

Theorem, CM: $\mathcal{O}(\mathcal{C}) = \mathcal{C}$.

This answers a question of Brian White.

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Strong rigidity in MCF

Blowing up sees compact pieces of the limit.

Is a piece enough to determine the limit?

Theorem, CIM, CM. *MCF shrinker close enough to a cylinder on a large central set must be a cylinder.*

What is needed for strong rigidity?

Extension: Still close on larger scale, loss in estimates.

Improvement: They are even closer on the larger scale.

Iterating these gives rigidity.

These are also the keys to uniqueness.

Flatness propagates

- **Flatness propagates forward in time, with a loss.**
- This is **Pseudo-locality**.
- Brakke in MCF; cf. Perelman in Ricci Flow.

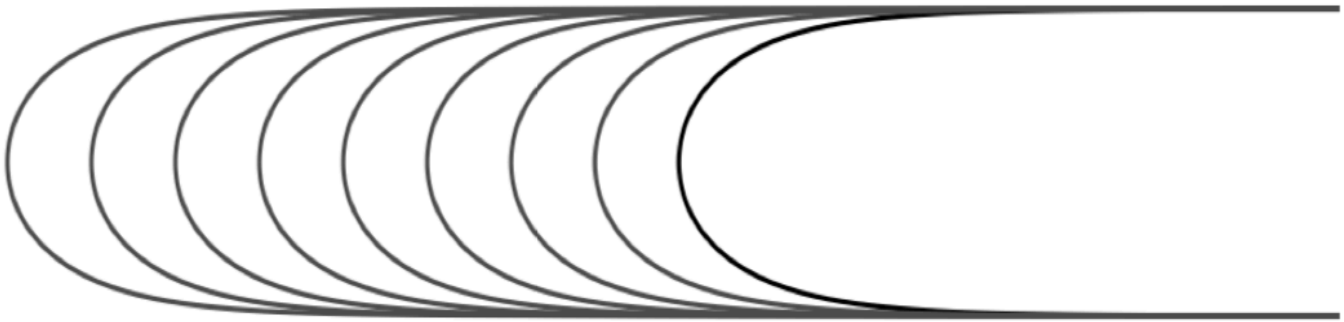


Figure: Grim reaper - stays flat, but not for long

Why can we extend?

Pseudo-locality: Estimates at one time give future estimates.

Scaling: For a shrinker, future $=$ larger scale.

This gives an extension, but with a loss in estimates.

Strong rigidity holds if we can recover this loss.

Quick Review: Hamilton's Ricci flow

- Ricci flow: $\partial_t g = -2\,\mathrm{Ric}_{g(t)}$.
- (M, g, f) is a gradient shrinking Ricci soliton (or *shrinker*) if

$$\mathrm{Ric} + \mathrm{Hess}_f = \frac{1}{2}\, g\,.$$

- Hessian $=$ Lie derivative of g, while $\frac{1}{2}\, g$ gives scaling: Shrinkers move by rescaling, up to diffeo's.

Blowing ups in Ricci flow

- Shrinkers as blowups using Perelman's monotonicities:
  - Reduced volume, Naber, Enders-Müller-Topping.
  - Shrinker entropy, Mantagazza-Müller.
- Subsequence converges to a limit **tangent flow**.
- Time-slice of the limit is a shrinker, typically non-compact.

Rigidity of cylinders

Theorem, CM: $\mathbf{S}^\ell \times \mathbf{R}^{n-\ell}$ strongly rigid for any ℓ.

Strong rigidity: If another shrinker is close enough on a large compact set, then it must agree.