Effective quasianalytic Remez and Łojasiewicz inequalities on tame sets

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Quasianalytic Remez inequalities

Theorem [Remez 1936]

Let $E \subseteq [0,1]$ be (Lebesgue) measurable with $|E| > 0$. Then

$$\|p\|_{[0,1]} \leq T_d\Big(\frac{2-|E|}{|E|}\Big)\,\|p\|_E$$

for all real polynomials of degree at most d, where $T_d(x) = \cos(d \arccos x)$ is the d-th Chebyshev polynomial.

Theorem [Brudnyi–Ganzburg 1973]

Let $K \subseteq \mathbb{R}^n$ be a convex body and $E \subseteq K$ measurable with $|E| > 0$. Then

$$\|p\|_K \leq T_d\Big(\frac{|K|^{1/n} + (|K|-|E|)^{1/n}}{|K|^{1/n} - (|K|-|E|)^{1/n}}\Big)\,\|p\|_E \leq \Big(\frac{4n\,|K|}{|E|}\Big)^d\,\|p\|_E$$

for all real polynomials of degree at most d.

Theorem [Pierzchała 2015]

Let $K \subseteq \mathbb{R}^n$ be a compact fat subanalytic set. Then there exist $\nu > 0$ and $m \in \mathbb{N}_{\geq 1}$ (depending only on K) such that

$$\|p\|_K \leq T_{md}\left(\frac{|K|^\nu + (|K| - |E|)^\nu}{|K|^\nu - (|K| - |E|)^\nu}\right) \|p\|_E \leq \left(\frac{4\,|K|}{\nu\,|E|}\right)^{md} \|p\|_E$$

for all measurable $E \subseteq K$ with $|E| > 0$ and all real polynomials of degree at most d.

Remark

The result holds for a large collection $\mathscr{K}(\mathbb{R}^n)$ of compact subsets of $\mathbb{R}^n$. For instance, $\mathscr{K}(\mathbb{R}^n)$ contains all compact UPC-sets, in particular, compact fat sets that are definable in the expansion of $\mathbb{R}$ by restricted quasianalytic Denjoy–Carleman functions.

The Remez inequality has been extended to various classes of functions: analytic, plurisubharmonic, holomorphic, exponential polynomial, ... by various authors.

We focus on quasianalytic Denjoy–Carleman classes.

Denjoy–Carleman classes $\mathcal{C}_M$

Let $M = (M_j)_{j\geq 1}$ be a positive sequence. For open $U \subseteq \mathbb{R}^n$, the class $\mathcal{C}_M(U)$ consists of all $f \in \mathcal{C}^\infty(U)$ such that for each compact $K \subseteq U$ there exist $C, \rho > 0$ such that

$$|\partial^\alpha f(x)| \leq C\,\rho^{|\alpha|}\,M_{|\alpha|}, \qquad \text{for all } x \in K,\ \alpha \in \mathbb{N}^n.$$

Examples

For $M = (j!^s)$ with $s \geq 1$, we get the Gevrey classes, in particular, the real analytic functions for $s = 1$.

Facts

Let us set $\mu_j := \frac{M_j}{M_{j-1}}$, for $j \geq 1$, so that $M_j = M_0 \mu_1 \mu_2 \cdots \mu_j$. We always assume $0 < \mu_1 \leq \mu_2 \leq \cdots$ and

$$\sum_{j \geq 1} \frac{1}{\mu_j} = \infty.$$

Equivalently, the class $\mathcal{C}_M$ is quasianalytic (i.e., for connected U and $x \in U$, the Taylor series homomorphism $T_x : \mathcal{C}_M(U) \to \mathbb{R}[\![x_1, \ldots, x_n]\!]$ is injective), by the Denjoy–Carleman theorem.

Under mild additional conditions, the expansion $\mathbb{R}_{\mathcal{C}_M}$ of $\mathbb{R}$ by all restricted $\mathcal{C}_M$-functions is a polynomially bounded o-minimal structure.
[Rolin–Speissegger–Wilkie 2003]

Examples

The Denjoy weights $\mu_j^s := j!(\log(ej))^s$, for $j \geq 1$ and $s \in [0, 1]$, are quasianalytic (and satisfy the mild additional conditions).

Bang degree

Let $\mu = (\mu_j)_{j\geq 1}$ be quasianalytic and $M_0, a, b > 0$ with $M_0 > b$. We define

$$\mathbf{N} = \mathbf{N}(\mu, M_0, a, b) := 2 \sup \left\{ n \in \mathbb{N} : \sum_{j=j_0(\frac{b}{M_0})+1}^{n} \frac{1}{\mu_j} < a\, e \right\}$$

where $j_0(\frac{b}{M_0}) := \lceil \log \frac{M_0}{b} \rceil_{\mathbb{N}}$. Then $\mathbf{N} \in \mathbb{N}_{\geq 2}$. This integer is a variant of a number introduced by [Bang 1953].

We also associate the constant $\mathbf{C_N} := (4e^{4+\gamma_{\tilde{\mu}}(\mathbf{N})})^{\mathbf{N}}$.

Here $\tilde{\mu} : [1, \infty) \to (0, \infty)$ is an increasing continuous and piecewise $\mathcal{C}^1$ function with $\mu_j = \tilde{\mu}(j)$ for $j \geq 1$, and

$$\gamma_{\tilde{\mu}}(n) := \sup_{1\leq s\leq n} \frac{s\tilde{\mu}'(s)}{\tilde{\mu}(s)}, \quad n \in \mathbb{N}_{\geq 1}.$$

Theorem [Nazarov–Sodin–Volberg 2004], [R 2023]

Let $\mu = (\mu_j)_{j\geq 1}$ be quasianalytic, $M_0 > b_0 > 0$, and $\mathbf{N} = \mathbf{N}(\mu, M_0, 1, b_0)$. Assume that $f : [0,1] \to \mathbb{R}$ satisfies $\|f\|_{[0,1]} \geq b_0$ and

$$\|f^{(j)}\|_{[0,1]} \leq M_j, \qquad \text{for } 0 \leq j \leq \mathbf{N}. \tag{1}$$

Then, for each interval $I \subseteq [0,1]$ and each measurable $E \subseteq I$ with $|E| > 0$,

$$\|f\|_I \leq \mathbf{C_N} \left(\frac{|I|}{|E|}\right)^{\mathbf{N}} \|f\|_E.$$

Remark (dependence on the size of f)

Consider $p_k(x) = \frac{x^k}{k!}$. Then $f_k := p_k|_{[0,1]}$ satisfies (1) for $M_j \equiv 1$ and all j. But $\|f_k\|_{[0,1]} = \frac{1}{k!}$ and $\|f_k\|_{[0,\frac{1}{2}]} = \frac{1}{2^k k!}$ so that a Remez inequality for f_k with constant independent of k cannot hold.

Theorem [R 2023]

Let $\mu = (\mu_j)_{j\geq 1}$ be quasianalytic and $M_0 > b_0 > 0$. Let $K \subseteq L \subseteq \mathbb{R}^n$ be convex bodies and $\mathbf{N} = \mathbf{N}(\mu, M_0, \operatorname{diam}(K), b_0)$. Assume that $f : L \to \mathbb{R}$ satisfies $\|f\|_K \geq b_0$ and

$$\|f\|_{j,L} := \sup_{x\in L} \sum_{|\alpha|=j} \frac{j!}{\alpha!} |\partial^\alpha f(x)| \leq M_j, \qquad \text{for } 0 \leq j \leq \mathbf{N}. \tag{2}$$

If $E \subseteq K$ is measurable with $|E| > 0$, then

$$\|f\|_K \leq \mathbf{C_N} \left(\frac{|K|^{1/n}}{|K|^{1/n} - (|K| - |E|)^{1/n}} \right)^{\mathbf{N}} \|f\|_E.$$

For later reference, we say that $f : L \to \mathbb{R}$ is $M^{|\mathbf{N}}$-smooth if (2) holds.

Main result

The quasianalytic Remez inequality extends to a large class of tame sets with effective constants depending on the geometry and the quasianalytic function.

Theorem 1 [R 2026]

Let $L \subseteq \mathbb{R}^n$ be a convex body and $K \in \mathscr{K}(\mathbb{R}^n)$ such that $K \subseteq L$. There exist $\nu_K \in (0,1]$ and $a_K > 0$ such that the following holds.

Let $\mu = (\mu_j)_{j\geq 1}$ be quasianalytic with $\mu_j \geq j$, for all $j \geq 1$, and $M_0 > b_0 > 0$. Let

$$\mathbf{N} = \mathbf{N}(\mu, M_0, a_K, b_0).$$

Assume that $f : L \to \mathbb{R}$ is $M^{|\mathbf{N}}$-smooth and satisfies $\|f\|_K \geq b_0$. If $E \subseteq K$ is measurable with $|E| > 0$, then

$$\|f\|_K \leq \mathbf{C_N} \left(\frac{|K|^{\nu_K}}{|K|^{\nu_K} - (|K| - |E|)^{\nu_K}} \right)^{\mathbf{N}} \|f\|_E.$$

The collection $\mathscr{K}(\mathbb{R}^n)$

A subset $K \subseteq \mathbb{R}^n$ belongs to $\mathscr{K}(\mathbb{R}^n)$ if there are exist $d \in \mathbb{N}_{\geq 1}$, $\delta_1, \delta_2, \delta_3 > 0$, a family $(K_w)_{w \in W}$ of convex bodies in $[0,1]^n$, and a family $(R_w)_{w \in W}$ of polynomial maps $\mathbb{R}^n \to \mathbb{R}^n$ of degree at most d such that

- $|K_w| \geq \delta_1$ for all $w \in W$,
- $\delta_2 \leq \| \operatorname{Jac}(R_w)\|_1 \leq \delta_3$ for all $w \in W$,
- $K = \bigcup_{w \in W} R_w(K_w)$.

The constant a_K

$$a_K := \begin{cases} 4\frac{(d+1)^{d+1}}{(d-1)^{d-1}} \max\{\operatorname{diam}(K), 1\}, & \text{if } d \geq 2, \\ \operatorname{diam}(K), & \text{if } d = 1. \end{cases}$$

Examples

For convex bodies, $d = 1$; for α-Hölder sets, $d = \lceil \alpha^{-1} \rceil$.

Application: Łojasiewicz inequalities

Thickness

A measurable set $K \subseteq \mathbb{R}^n$ is called (c_K, θ_K)-thick if there exist $c_K, \theta_K > 0$ such that, for all $x \in K$ and $0 < r \leq 1$, $|K \cap B(x,r)| \geq c_K\, r^{\theta_K}$.

Theorem 2 [R 2026]

In the setting of Theorem 1, assume that K is (c_K, θ_K)-thick. Let $\mathbf{N} = \mathbf{N}(\mu, M_0, a_K, b_0)$. Let $f : L \to \mathbb{R}$ be $M^{|\mathbf{N}}$-smooth and satisfy $\|f\|_K \geq b_0$. For any function $g : K \to [-1,1]$,

$$\|f\|_{B(x, \frac{|g(x)|}{2}) \cap K} \geq \frac{\|f\|_K}{\mathbf{C_N}} \left(\frac{c_K\, \nu_K}{2^{\theta_K} |K|} \right)^{\mathbf{N}} |g(x)|^{\theta_K \mathbf{N}}, \quad x \in K.$$

Corollary

Take $g(x) = \min\{\mathrm{dist}(x, f|_K^{-1}(0)), 1\}$ or $g \equiv r$ for $r \in (0,1)$:

$$\mathrm{ord}_x\, f = \limsup_{r \to 0} \frac{\log \|f\|_{B(x,r) \cap K}}{\log r} \leq \theta_K \mathbf{N}.$$

The inequality is trivial if $x \in g^{-1}(0)$. Fix $x \in K \setminus g^{-1}(0)$. Set $r = |g(x)|$ and $B = B(x, \frac{r}{2}) \cap K$. The Remez inequality implies the sublevel estimate

$$|K_t| \le \frac{|K|}{\nu_K} \Big(\frac{\mathbf{C_N}\, t}{\|f\|_K} \Big)^{1/\mathbf{N}}, \quad t > 0, \tag{3}$$

for $K_t := \{x \in K : |f(x)| \le t\}$.

Let T be the set of all $t > 0$ such that RHS of (3) is $< c_K (\frac{r}{2})^{\theta_K}$, i.e., $t \in T$ iff

$$0 < t < \frac{\|f\|_K}{\mathbf{C_N}} \Big(\frac{c_K\, \nu_K}{2^{\theta_K} |K|} \Big)^{\mathbf{N}} r^{\theta_K \mathbf{N}}.$$

Since $|B| \ge c_K (\frac{r}{2})^{\theta_K}$ because K is (c_K, θ_K)-thick, we have $B \not\subseteq K_t$ for all $t \in T$. Consequently, $\|f\|_B > t$ for all $t \in T$. The assertion follows.

Theorem 3 [R 2026]

Adopt the setting of Theorem 2 and assume that $M = (M_j)$ is such that $\mathbb{R}_{\mathcal{C}_M}$ is an o-minimal structure. Let $f : L \to \mathbb{R}$ be $M^{|\mathbf{N}}$-smooth and satisfy $\|f\|_K \geq b_0$. Assume $(\operatorname{grad} f|_K)^{-1}(0) \subseteq f|_K^{-1}(0)$. Then:

- in a neighborhood of $f|_K^{-1}(0)$ in K,

$$|\operatorname{grad} f(x)| \geq \frac{1}{2\sqrt{M_2}} \left(\frac{\|f\|_K}{2\mathbf{C_N}}\right)^{\frac{1}{\theta_K \mathbf{N}}} \left(\frac{c_K\, \nu_K}{|K|}\right)^{\frac{1}{\theta_K}} |f(x)|^{\frac{3}{2} - \frac{1}{\theta_K \mathbf{N}}};$$

- in a neighborhood of $f|_K^{-1}(0)$ in K,

$$\|\operatorname{grad} f(x)\|_{B_x} \geq \frac{1}{2} \left(\frac{2\|f\|_K}{3\mathbf{C_N}}\right)^{\frac{1}{\theta_K \mathbf{N}}} \left(\frac{c_K\, \nu_K}{|K|}\right)^{\frac{1}{\theta_K}} |f(x)|^{1 - \frac{1}{\theta_K \mathbf{N}}},$$

  where B_x is a neighborhood of x in L that shrinks to x if $x \to f|_K^{-1}(0)$.

Based on

- A. Rainer, *Quantitative tame properties of differentiable functions with controlled derivatives*, Nonlinear Analysis **237** (2023), 113372, doi:10.1016/j.na.2023.113372.
- A. Rainer, *Effective quasianalytic Remez inequalities on tame sets*, arXiv:2606.08183.

Thank you for your attention!