Perturbation of Polynomials and Linear Operators

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Based on joint work with Armin Rainer



Regularity problem

Consider a monic polynomial of degree d,

$$P_a(Z) = Z^d + \sum_{j=1}^{d} a_j Z^{d-j},$$

where $a = (a_1, \ldots, a_d) : I \to \mathbb{C}^d$ and $I \subseteq \mathbb{R}$ is an interval. Assuming that a is smooth, i.e. C^∞, how regular can the roots be?

In general, the roots cannot be Lipschitz: $Z^2 = x$.

Spagnolo's conjecture (2000)

The roots can be represented by absolutely continuous functions.

The conjecture was proved in [P–Rainer 2016] and the optimal regularity was established in [P–Rainer 2018].

Theorem [P–Rainer 2018]

Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let P_a be a monic polynomial of degree d with coefficient vector $a \in C^{d-1,1}(\overline{I}, \mathbb{C}^d)$. If $\lambda \in C^0(I)$ is a root of P_a, then $\lambda \in W^{1,q}(I)$ for each $q \in [1, \frac{d}{d-1})$ and

$$\|\lambda'\|_{L^q(I)} \leq C(d,q)\, \max\{1, |I|^{1/q}\} \max_{1 \leq j \leq d} \|a_j\|^{1/j}_{C^{d-1,1}(\overline{I})}.$$

Remarks

- Continuous root systems exist (in the one-parameter case).
- The multiparameter case follows.
- Optimality: examples show that the assumption $a \in C^{d-1,1}$ cannot be relaxed and $\lambda' \in L^{d/(d-1)}$ cannot be expected.
- Stability?

For $p = \frac{d}{d-1}$: $\quad t^{1/d} \notin W^{1,p}$ but $t^{1/d} \in W^{1,q}$ for all $q < p$.

- Radical case: $f = g^{1/d}$, [Ghisi-Gobbino 2013]: Higher Glaeser inequalities.
- [P–Rainer 2016] For the space M of coefficients $a_0, \dots a_{d-1}$ construct the tower of principalizations of the generalized discriminant ideals $\mathcal{D}_m$

$$\mathbb{C}^d \xleftarrow{\sigma_2} M_2 \xleftarrow{\sigma_{3,2}} M_3 \xleftarrow{\sigma_{4,3}} \cdots \xleftarrow{\sigma_{d,d-1}} M_d.$$

  On M_d we have formulas for the roots

$$\sum_{m=1}^{d} A_m \, f_m^{\alpha_m} \psi_m(y_{m,1}^{1/q_m}, \dots, y_{m,r_m}^{1/q_m}, y_{m,r_m+1} \dots, y_m)$$

  Here $A_M \in \mathbb{Q}$, $f_m \in \mathscr{D}_m$ is a local generator of $\sigma_m^*(\mathscr{D}_m)$, ψ_m is a convergent power series, $q_m \in \mathbb{N}_{\geq 1}$, $\alpha_m \in \frac{1}{q_m}\mathbb{N}_{\geq 1}$, and $y_{m,i}$ form a local system of coordinates on M_m.
- These formulas reduce the problem to radicals for the curves that can be lifted to M_d. Then approximate the given curve of coefficients by polynomial curves that can be lifted.

- Radical case: $f = g^{1/d}$, [Ghisi-Gobbino 2013]. Higher Glaeser inequalities.
- [P–Rainer 2018] Near a s.t. P_a has at least two distinct roots P_a factors out

$$P_a = Z^d + \sum_{j=1}^{d} a_j Z^{d-j} = (Z^{d_1} + \sum_{j=1}^{d_1} b_j Z^{d_1-j})(Z^{d_2} + \sum_{j=1}^{d_2} c_j Z^{d_2-j})$$

  Lemma. [IFT] Let $P_a = P_b P_c$, where P_b and P_c are monic real polynomials without common root. Then $P = P_b P_c$ for an analytic mapping $a \mapsto (b(a), c(a))$.
  (Lemma can be used to show classical Puiseux).
- Use Lemma to factorize $P_{a(t)}$ on (possibly infinte number of) subintervals. Proceed by induction on the degree d with some more explicit assumptions on the coefficients. They are preserved by factorisation by the radical case.
- If there is a single root one uses the radical case.

Unordered d-tuples of complex numbers

We work with unordered d-tuples $[z_1, \dots, z_d] = \sum_{i=1}^{d} \delta_{z_i}$, for $z_i \in \mathbb{C}$. Then $\mathcal{A}_d(\mathbb{C}) = \{[z] : z \in \mathbb{C}^d\} = \mathbb{C}^d / \mathrm{S}_d$ endowed with

$$\mathbf{d}([z],[w]) = \min_{\sigma \in \mathrm{S}_d} \|z - \sigma w\|_2$$

is a complete metric space. ($\frac{\mathbf{d}}{\sqrt{d}}$ induced by L^2-Wasserstein metric on $\mathscr{P}(\mathbb{C})$.)

d-valued Sobolev functions [Almgren 2000]

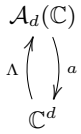
There is a bi-Lipschitz embedding $\Delta : \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^N$, $N = N(d)$. Define

$$W^{1,q}(U, \mathcal{A}_d(\mathbb{C})) := \{f : U \to \mathcal{A}_d(\mathbb{C}) : \Delta \circ f \in W^{1,q}(U, \mathbb{R}^N)\}$$

and equip it with the complete metric

$$(f, g) \mapsto \|\Delta \circ f - \Delta \circ g\|_{W^{1,q}(U,\mathbb{R}^N)}.$$

For $1 \leq q < \infty$, the induced topology is independent of Δ.



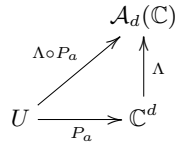
The elementary symmetric functions induce a bijection $a : \mathcal{A}_d(\mathbb{C}) \to \mathbb{C}^d$. Let $\Lambda = a^{-1}$ be its inverse. Then Λ assigns to P_a its unordered d-tuple of roots. Note that a is locally Lipschitz and Λ is locally $\frac{1}{d}$-Hölder.

The solution map

By the optimal regularity result,

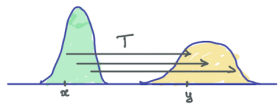
$$\mathcal{S} = \Lambda_* : C^{d-1,1}(U, \mathbb{C}^d) \to W^{1,q}_{\mathrm{loc}}(U, \mathcal{A}_d(\mathbb{C})),$$
$$(x \mapsto P_{a(x)}) \mapsto (x \mapsto \Lambda(P_{a(x)}))$$

is well-defined and bounded, for $q \in [1, \frac{d}{d-1})$.



- Monge problem: Given probabilty measures $\mu \in \mathscr{P}(X)$, $\nu \in \mathscr{P}(Y)$ and a cost function $c(x, y)$ find $T : X \to Y$, $T_{\#}(\mu) = \nu$, minimizing

  Total cost $= \int c(x, T(x))\, d\mu(x)$



- Kantorovich problem:

$$\inf \int_{X \times Y} c(x, y)\, d\gamma(x, y), \quad \text{for } (\pi_X)_{\#}(\gamma) = \mu, (\pi_Y)_{\#}(\gamma) = \nu.$$

- Let M be a Riemannian manifold. The q-Wasserstein distance beetween $\mu \in \mathscr{P}(M)$ and $\nu \in \mathscr{P}(M)$

$$W_q(\mu, \nu) = \inf_{\xi \in \Pi(\mu,\nu)} \Big(\int_{M \times M} \mathrm{dist}^q(x, y)\, d\xi(x, y) \Big)^{\frac{1}{q}},$$

  where $\Pi(\mu, \nu) = \{\xi \in \mathscr{P}(M \times M) | s.t. (\pi_1)_{\#}(\xi) = \mu, (\pi_2)_{\#}(\xi) = \nu\}$.

Univariate polynomials:

Solution map $\Lambda = [\lambda_1, \ldots, \lambda_d] : \mathbb{C}^d \to \mathcal{A}_d(\mathbb{C})$ can be understood as a map to the space of probability measures $\mathscr{P}_q(\mathbb{C})$

$$\mu = \frac{1}{d}\sum_{j=1}^{d} \delta_{\lambda_j} : \mathbb{C}^d \to \mathscr{P}_q(\mathbb{C}).$$

The q-Wasserstein distance beetween $\mu = \frac{1}{d}\sum_{j=1}^{d} \delta_{z_j}$, $\nu = \frac{1}{d}\sum_{j=1}^{d} \delta_{w_j}$

$$W_q(\mu, \nu) = (\min_{\sigma \in \mathrm{S}_d} \tfrac{1}{d}\|z - \sigma w\|^q)^{\frac{1}{q}} = \tfrac{1}{\sqrt[q]{d}}\mathbf{d}_q([z], [w])$$

Multivariate construction [Antonini–Cavalletti–Lerario 2023]

- $H_{n,d}$ is the space of complex homogeneous polynomials of degree d in $n+1$ variables, and $P_{n,d} = \mathbb{P}(H_{n,d})$. A continuous injective "root map" $\mu : P_{n,d} \to \mathscr{P}_q(\mathbb{CP}^n)$ is constructed. For $n > 1$, the image $\mu(P_{n,d})$ is not geodesically convex.
- Therefore the inner Wasserstein distance W_q^{in} is studied.
- In particular, the metric space $(P_{n,d}, W_2^{\mathrm{in}})$ is complete and geodesic.
- Theorem. For each $n \in \mathbb{N}$ there exists $e = e(n) > 0$ such that for every $P \in C^{ed-1,1}(I, P_{n,d})$ we have $\mu \circ P \in AC^q(I, \mathscr{P}_q(\mathbb{CP}^n))$, for all $q \in [1, \frac{ed}{ed-1})$.
  Proof Application of optimal Sobolev regularity.
- As a consequence, $(P_{n,d}, W_q^{\mathrm{in}})$ is compact, for all $q \in [1, \frac{ed}{ed-1})$.

Open problem: Is $(P_{n,d}, W_2^{\mathrm{in}})$ is compact?

Continuity of the solution map (motivated by a question of A. Lerario)

Theorem [P–Rainer 2024]

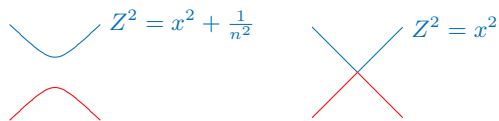
$\mathcal{S} : C^d(U, \mathbb{C}^d) \to W^{1,q}_{\mathrm{loc}}(U, \mathcal{A}_d(\mathbb{C}))$ is continuous, for all $q \in [1, \frac{d}{d-1})$.
In general, *not* uniformly continuous.

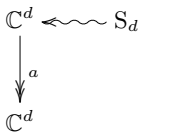
Consequences

Let $a_n \to a$ in $C^d(\overline{I}, \mathbb{C}^d)$. There exist continuous parameterizations $\lambda, \lambda_n : I \to \mathbb{C}^d$ of the roots of P_a, P_{a_n}, respectively. For all $q \in [1, \frac{d}{d-1})$:

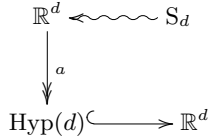
- L^q-convergence of metric speeds: $\big\| \|\lambda'\|_2 - \|\lambda_n'\|_2 \big\|_{L^q(I)} \to 0$.
- Convergence of q-energy: $\|\lambda_n'\|_{L^q(I,\mathbb{C}^d)} \to \|\lambda'\|_{L^q(I,\mathbb{C}^d)}$.
- If $\lambda_n \to \lambda$ in $C^0(\overline{I}, \mathbb{C}^d)$, then $\|\lambda' - \lambda_n'\|_{L^q(I,\mathbb{C}^d)} \to 0$.

Not every continuous system of roots is such a limit:





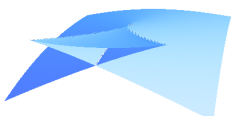
general case



hyperbolic case

The space of hyperbolic polynomials

A polynomial $P_a \in \mathbb{R}[Z]$ is called hyperbolic if all its roots are real. The space $\mathrm{Hyp}(d)$ of hyperbolic polynomials of degree d can be identified with a proper semialgebraic subset of $\mathbb{R}^d$.



The set of hyperbolic polynomials of the form $Z^4 + a_2 Z^2 + a_3 Z + a_4$ is the closure of the chamber in the middle.

The solution map

The roots of $P_a \in \mathrm{Hyp}(d)$ can be ordered, $\lambda_1^\uparrow \le \cdots \le \lambda_d^\uparrow$. This defines a continuous map $\lambda^\uparrow : \mathrm{Hyp}(d) \to \mathbb{R}^d$ (a right-inverse of $a : \mathbb{R}^d \to \mathrm{Hyp}(d)$).

It induces the solution map

$$\mathcal{S}^\uparrow = (\lambda^\uparrow)_* : C^{d-1,1}(U, \mathrm{Hyp}(d)) \to C^{0,1}(U, \mathbb{R}^d)$$

which is well-defined and bounded.

Bronshtein's theorem [Bronshtein 1979], [P–Rainer 2015]

Let $I \subseteq \mathbb{R}$ be an open interval. Let P_a be a hyperbolic polynomial of degree d with coefficient vector $a \in C^{d-1,1}(I, \mathbb{R}^d)$. If $\lambda \in C^0(I)$ is a root of P_a, then $\lambda \in C^{0,1}(I)$ and for any pair of relatively compact open intervals $I_0 \Subset I_1 \Subset I$,

$$\mathrm{Lip}_{I_0}(\lambda) \le C \max_{1 \le j \le d} \|a_j\|^{1/j}_{C^{d-1,1}(\overline{I}_1)},$$

where $C = C(d)\ \max\{\mathrm{dist}(I_0, \mathbb{R} \setminus I_1)^{-1}, 1\}$.

$\mathcal{S}^{\uparrow} : C^{d-1,1}(U, \mathrm{Hyp}(d)) \to C^{0,1}(U, \mathbb{R}^d)$ is not continuous

For $x^2 + \frac{1}{n^2} \to x^2$ as $n \to \infty$, $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$, and $f(x) = |x|$,

$$\mathrm{Lip}_{[0,1]}(f - f_n) \geq \left| \frac{(f - f_n)(\frac{1}{n}) - (f - f_n)(0)}{\frac{1}{n}} \right| = \left| \frac{\frac{1}{n} - \frac{\sqrt{2}}{n} + \frac{1}{n}}{\frac{1}{n}} \right| = 2 - \sqrt{2}.$$

Theorem [P–Rainer 2024]

$\mathcal{S}^{\uparrow} : C^d(U, \mathrm{Hyp}(d)) \to W^{1,q}_{\mathrm{loc}}(U, \mathbb{R}^d)$ is continuous, for all $q \in [1, \infty)$.

Corollary

$\mathcal{S}^{\uparrow} : C^d(U, \mathrm{Hyp}(d)) \to C^{0,\alpha}(U, \mathbb{R}^d)$ is continuous, for all $\alpha \in (0, 1)$.

In general, these maps are *not* uniformly continuous.

Corollary

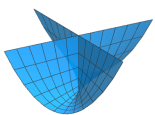
Let $U \subseteq \mathbb{R}^m$ be open and $U_0 \Subset U$. Let $a_n \to a$ in $C^d(U, \mathrm{Hyp}(d))$. We have:

- Convergence of surface area of graphs of the roots: for all $1 \leq j \leq d$,

$$\mathcal{H}^m(\mathrm{graph}_{U_0}(\lambda_j^{\uparrow} \circ a_n)) \to \mathcal{H}^m(\mathrm{graph}_{U_0}(\lambda_j^{\uparrow} \circ a)).$$

- Lower semicontinuity of area of zero sets: $\liminf_{n\to\infty} \mathcal{H}^m(Z_n) \geq \mathcal{H}^m(Z)$, where $Z = \{(x, y) \in U_0 \times \mathbb{R} : P_{a(x)}(y) = 0\}$ and Z_n is defined analogously.

Without hyperbolicity we cannot expect semicontinuity:



Let Z_t denote the intersection of the Whitney umbrella $\{x^2 - y^2 z = 0\}$ with the plane $\{z = t\}$ and the cylinder $\{x^2 + y^2 < 1\}$. Then $\mathcal{H}^1(Z_t) = 0$ if $t < 0$, $\mathcal{H}^1(Z_0) = 2$, and $\mathcal{H}^1(Z_t) = 4$ if $t > 0$.

The characteristic map

Consider the real vector space $\mathrm{Herm}(d) = \{A \in M_d(\mathbb{C}) : A^* = A\}$. Ordering the eigenvalues increasingly, defines a map $\lambda^{\uparrow} : \mathrm{Herm}(d) \to \mathbb{R}^d$.

It induces the characteristic map

$$\mathcal{E}^{\uparrow} = (\lambda^{\uparrow})_* : W^{1,q}(U, \mathrm{Herm}(d)) \to W^{1,q}(U, \mathbb{R}^d)$$

which is well-defined and bounded, for all $q \in [1, \infty]$.

Theorem [Löwner 1934]

For $A, B \in \mathrm{Herm}(d)$, we have $\|\lambda^{\uparrow}(A) - \lambda^{\uparrow}(B)\|_2 \leq \|A - B\|_2$.

Theorem [P–Rainer 2026]

$\mathcal{E}^{\uparrow} : W^{1,q}(U, \mathrm{Herm}(d)) \to W^{1,q}(U, \mathbb{R}^d)$ is continuous, for all $q \in [1, \infty)$.

Corollary

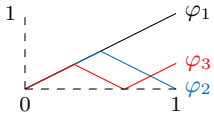
Let U be a Lipschitz domain. Then $\mathcal{E}^{\uparrow} : C^{0,1}(\overline{U}, \mathrm{Herm}(d)) \to C^{0,\alpha}(\overline{U}, \mathbb{R}^d)$ is continuous, for all $\alpha \in (0, 1)$, and discontinuous for $\alpha = 1$; e.g. $\begin{pmatrix} \frac{1}{n} & x \\ x & -\frac{1}{n} \end{pmatrix}$.

Theorem

If $A_n \to A$ in $C^{0,1}(U, \mathrm{Herm}(d))$ then $\nabla(\mathcal{E}^{\uparrow}(A_n)) \to \nabla(\mathcal{E}^{\uparrow}(A))$ a.e. in U.

Example

The nonnegative eigenvalues a_n, b_n of $\begin{pmatrix} \frac{1}{n} & \varphi_n \\ \varphi_n & -\frac{1}{n} \end{pmatrix}$, $\begin{pmatrix} \frac{1}{2n} & \varphi_n \\ \varphi_n & -\frac{1}{2n} \end{pmatrix}$ satisfy $\|a_n' - b_n'\|_{L^q([0,1])} \geq \frac{1}{12}$.



Example

Let $\alpha \in (0,1)$ and $r = \frac{\alpha}{1-\alpha}$. The nonnegative eigenvalues a_n, b_n of $\begin{pmatrix} \frac{1}{n^r} & nx \\ nx & -\frac{1}{n^r} \end{pmatrix}$, $\begin{pmatrix} \frac{1}{2n^r} & nx \\ nx & -\frac{1}{2n^r} \end{pmatrix}$ satisfy $|a_n - b_n|_{C^{0,\alpha}([0,1])} \geq \frac{1}{5}$.

Problem

Find effective moduli of continuity for the restrictions of the characteristic map to interesting compact subspaces.

The characteristic map

$\mathrm{Norm}(d) = \{A \in M_d(\mathbb{C}) : A^*A = AA^*\}$ is a singular real algebraic cone. The eigenvalue map $\Lambda : \mathrm{Norm}(d) \to \mathcal{A}_d(\mathbb{C})$ induces the characteristic map

$$\mathcal{E} = \Lambda_* : W^{1,q}(U, \mathrm{Norm}(d)) \to W^{1,q}(U, \mathcal{A}_d(\mathbb{C}))$$

which is well-defined and bounded, for all $q \in [1, \infty]$.

Theorem [Hoffman–Wielandt 1953]

For $A, B \in \mathrm{Norm}(d)$, we have $\mathbf{d}(\Lambda(A), \Lambda(B)) \leq \|A - B\|_2$.

Theorem [P–Rainer 2026]

$\mathcal{E} : W^{1,q}(U, \mathrm{Norm}(d)) \to W^{1,q}(U, \mathcal{A}_d(\mathbb{C}))$ is continuous, for all $q \in [1, \infty)$.

No corresponding results for the eigenvectors (beyond quasianalyticity)

There is no continuous orthonormal frame of eigenvectors for the symmetric matrix $e^{-1/x^2} \begin{pmatrix} \cos x^{-1} & \sin x^{-1} \\ \sin x^{-1} & -\cos x^{-1} \end{pmatrix}$.

Singular values

The singular values of $A \in M_{D,d}(\mathbb{C})$, $d \leq D$, are the nonnegative square roots of the eigenvalues of A^*A,

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_d \geq 0.$$

They define a Lipschitz map $\sigma : M_{D,d}(\mathbb{C}) \to \mathbb{R}^d$.

The Hermitian matrix $\begin{pmatrix} 0 & \tilde{A} \\ \tilde{A}^* & 0 \end{pmatrix}$, where $\tilde{A}$ results from A by adding $D - d$ zero columns, has the eigenvalues $\sigma_1, \ldots, \sigma_d, 0, \ldots, 0, -\sigma_d, \ldots, -\sigma_1$.

Corollary

The map

$$\sigma_* : W^{1,q}(U, M_{D,d}(\mathbb{C})) \to W^{1,q}(U, \mathbb{R}^d)$$

is bounded, for all $q \in [1, \infty]$, and continuous, for all $q \in [1, \infty)$.

Corollary

If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz, then the composite

$$W^{1,q}(U, M_{D,d}(\mathbb{C})) \xrightarrow{(\sigma_i)_*} W^{1,q}(U, \mathbb{R}) \xrightarrow{f_*} W^{1,q}(U, \mathbb{R})$$

is continuous, for all $q \in [1, \infty)$. [Marcus–Mizel 1979]

Condition number of $A \in M_d(\mathbb{C})$

$\kappa(A) = \|A^{-1}\|_{\mathrm{op}} \|A\|_{\mathrm{op}} = \frac{\sigma_1(A)}{\sigma_d(A)}$

Corollary

Let $q \in [1, \infty)$. Let $A_0 \in W^{1,q}(I, M_d(\mathbb{C}))$ with $\inf_{x \in I} \sigma_d(A_0(x)) > 0$. Then the condition number $\kappa(A) \in W^{1,q}(I)$ is well-defined in a neighborhood $\mathcal{U}$ of A_0 in $W^{1,q}(I, M_d(\mathbb{C}))$ and $\kappa : \mathcal{U} \to W^{1,q}(I)$ is continuous.

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Thank you for your attention!